HOMEWORK 7

Due in class *Wed, Apr. 8.*

1. *Lee, Problem 5-6.* Suppose $M \subset \mathbb{R}^n$ is an embedded m-dimensional submanifold, and let $UM \subset T\mathbb{R}^n$ be the set of all unit tangent vectors to M:

$$UM := \{(x, v) \in T\mathbb{R}^n \mid x \in M, v \in T_x M, |v| = 1\}.$$

It is called the *unit tangent bundle of* M. Prove that UM is an embedded $(2m-1)$-dimensional submanifold of $T\mathbb{R}^n \approx \mathbb{R}^n \times \mathbb{R}^n$.

2. *Lee, Problem 6-2.* Prove the *Whitney immersion theorem*: every smooth n-manifold admits a smooth immersion into $\mathbb{R}^{2n}$.

Hint: using the Whitney embedding theorem, we can assume without loss of generality that M is an embedded n-dimensional submanifold of $\mathbb{R}^{2n+1}$. Let $UM \subset T\mathbb{R}^{2n+1}$ be the unit tangent bundle of M, and let $G\colon UM \to \mathbb{R}P^{2n}$ be the map $G(x, v) = [v]$. Use Sard's theorem to conclude that there is some $v \in \mathbb{R}^{2n+1} \setminus \mathbb{R}^{2n}$ such that $[v]$ is not in the image of G, and show that the projection from $\mathbb{R}^{2n+1}$ to $\mathbb{R}^{2n}$ with kernel $\mathbb{R}v$ restricts to an immersion of M into $\mathbb{R}^{2n}$.

3. *Lee, Problem 7-1.* Show that for any Lie group G, the multiplication map $m\colon G \times G \to G$ is a smooth submersion. *Hint: use local sections*

4. *Lee, Problem 7-4.* Let $\det\colon GL(n, \mathbb{R}) \to \mathbb{R}$ be the determinant function. Use Corollary 3.25 to compute the differential of det, as follows.

(a) For any $A \in M(n, \mathbb{R})$, show that

$$\left.\frac{d}{dt}\right|_{t=0} \det(I_n + tA) = \operatorname{Tr} A.$$

Hint: Writing determinant as a sum over permutations, $\det(I_n + tA)$ *is a polynomial in* t*. What is the linear term?*

(b) For $X \in GL(n, \mathbb{R})$ and $B \in T_X GL(n, \mathbb{R}) \cong M(n, \mathbb{R})$, show that

$$d(\det)_X(B) = (\det X) \operatorname{Tr}(X^{-1}B).$$

Hint: $\det(X + tB) = \det(X) \det(I_n + tX^{-1}B)$.

5. *Lee, Problem 7-11.* Considering S^{2n+1} as the unit sphere in $\mathbb{C}^{n+1}$, define an action of S^1 on S^{2n+1}, called the *Hopf action*, by

$$z \cdot (w^1, \dots, w^{n+1}) = (zw^1, \dots, zw^{n+1}).$$

Show that this action is smooth and its orbits are disjoint unit circles in $\mathbb{C}^{n+1}$ whose union is S^{2n+1}.