HOMEWORK 9

Due in class *Mon, May 4*.

1. Let V be a finite-dimensional vector space. Recall that given two tensors $\omega \in T^k V^*$ and $\eta \in T^\ell V^*$, the tensor product $\omega \otimes \eta \in T^{k+\ell} V^*$ is defined by

$$\omega \otimes \eta(v_1, \dots, v_k, v_{k+1}, \dots, v_{k+\ell}) = \omega(v_1, \dots, v_k)\eta(v_{k+1}, \dots, v_{k+\ell}).$$

Let $\det \in T^2(\mathbb{R}^2)^*$ be the 2-tensor defined by $\det(v, w) = \det \left(\begin{smallmatrix} v^1 & w^1 \\ v^2 & w^2 \end{smallmatrix}\right) = v^1 w^2 - v^2 w^1$, where $v = v^i E_i$ and $w = w^i E_i$. Recall that if (e^1, e^2) is the standard basis for $(\mathbb{R}^2)^*$, then $\det = e^1 \otimes e^2 - e^2 \otimes e^1$. Determine (with proof) whether or not there are 1-tensors (covectors) $\omega, \eta \in T^1(\mathbb{R}^2)^* = (\mathbb{R}^2)^*$ such that $\det = \omega \otimes \eta$.

2. *Lee, Problem 12-7.* Let (e^1, e^2, e^3) be the standard dual basis for $(\mathbb{R}^3)^*$. Show that $e^1 \otimes e^2 \otimes e^3$ is not equal to a sum of an alternating tensor and a symmetric tensor.

3. Let A be a positive definite symmetric 2×2 matrix, and let $\langle \cdot, \cdot \rangle$ be the inner product it induces on $\mathbb{R}^2$ by

$$\langle v, w \rangle = [v^1 \; v^2] A \begin{bmatrix} w^1 \\ w^2 \end{bmatrix} \text{ for } v, w \in \mathbb{R}^2.$$

Recall that the musical isomorphism $\flat \colon \mathscr{X}(\mathbb{R}^2) \to \mathscr{X}^*(\mathbb{R}^2)$ is defined by $X_p^\flat(v) = \langle X_p, v \rangle$ for $X \in \mathscr{X}(\mathbb{R}^2)$, $p \in \mathbb{R}^2$, and $v \in T_p\mathbb{R}^2 = \mathbb{R}^2$. Let $X \in \mathscr{X}(\mathbb{R}^2)$ be given by $X = X^i \frac{\partial}{\partial x^i}$ relative to the coordinate frame for $T\mathbb{R}^2$, and let its associated covector field $X^\flat \in \mathscr{X}^*(\mathbb{R}^2)$ be given by $X^\flat = X_i dx^i$ relative to the coordinate frame for $T^*\mathbb{R}^2$. Find an expression for X_i in terms of X^i and the matrix A.

4. Let $\omega^1, \dots, \omega^k$ be covectors on a finite-dimensional vector space V.

(a) *Lee, Problem 14-1.* Show that $\omega^1, \dots, \omega^k$ are linearly dependent if and only if $\omega^1 \wedge \cdots \wedge \omega^k = 0$.

(b) Suppose $\omega^1, \dots, \omega^k$ are linearly independent, and so is the collection of covectors $\eta^1, \dots, \eta^k \in V^*$. Prove that $\operatorname{span}(\omega^1, \dots, \omega^k) = \operatorname{span}(\eta^1, \dots, \eta^k)$ if and only if there is some nonzero real number c such that $\omega^1 \wedge \cdots \wedge \omega^k = c\, \eta^1 \wedge \cdots \wedge \eta^k$.

5. *Lee, Problem 14-7, (a) and (b).* In each of the following, M and N are smooth manifolds; $F\colon M \to N$ is smooth; and ω is a smooth differential form on N. In each case, compute $d\omega$ and $F^*\omega$, and verify by direct computation that $F^*(d\omega) = d(F^*\omega)$.

(a) $M = N = \mathbb{R}^2$;
$F(s,t) = (st, e^t)$;
and $\omega = x\,dy$.

(b) $M = \mathbb{R}^2$ and $N = \mathbb{R}^3$;
$F(\theta,\varphi) = \big((\cos\varphi + 2)\cos\theta,\ (\cos\varphi + 2)\sin\theta,\ \sin\varphi\big)$;
$\omega = y\,dz \wedge dx$.

6. *Lee, Problem 16-2.* Let $\mathbb{T}^2 = S^1 \times S^1 \subset \mathbb{R}^4$ denote the 2-torus, defined as the set of points (w,x,y,z) such that $w^2 + x^2 = y^2 + z^2 = 1$, with the product orientation determined by the standard orientation on S^1. Consider the 2-form on $\mathbb{R}^4$ given by

$$\omega = xyz\,dw \wedge dy \in \Omega^2(\mathbb{R}^4),$$

and compute $\int_{\mathbb{T}^2} \omega$.