The Jordan-Schönflies Theorem and the Classification of Surfaces

Carsten Thomassen

INTRODUCTION. The Jordan curve theorem says that a simple closed curve in the Euclidean plane partitions the plane into precisely two parts: the interior and the exterior of the curve. Although this fundamental result seems intuitively obvious it is fascinatingly difficult to prove. There are several proofs in the literature. For example, Tverberg [**12**] gave a proof involving only approximation with polygons. Here, we give a short proof based only on a trivial part of Kuratowski's theorem on graph planarity (see Lemma 2.5, below), namely, that $K_{3,3}$ is not planar.

Then we turn to another fundamental topological result: the classification of (compact) surfaces. A *surface* is a connected compact topological space which is locally homeomorphic to a disc (that is, the interior of a circle in the plane). The classification of surfaces says that every surface is homeomorphic to a space obtained from a sphere by adding handles or crosscaps. One of the first complete proofs was given by Kerékjártó [**4**] and there are several short proofs based on the assumption that every surface can be triangulated (see e.g. [**1, 2**]). Tutte [**11**] gave a proof in a purely combinatorial framework. In this paper we present a self-contained proof. The proof consists of two parts: a "topological" part and a "combinatorial" part. The combinatorial part (Section 5) is very short. It differs from other proofs in that it uses no topological results, not even the Jordan curve theorem. In particular, it does not use Euler's formula (which includes the Jordan curve theorem). Thus, the combinatorial part can be read independently of the previous results and it is of interest to those applications (for example to the Heawood problem mentioned below) where the surfaces under consideration are already triangulated.

The topological part is a proof of the fact that every surface S can be triangulated, i.e., S is homeomorphic to a topological space obtained by pasting triangles together. The idea behind this is simple: First we consider, for each point p in S, a small disc D_p around p. As S is compact, S is covered by a finite collection of the discs D_p. If S minus the boundaries of those discs consists of a finite number of connected components, then each of these is homeomorphic to a disc and it is then easy to triangulate S. However, the discs D_p may overlap in a complicated way. The previous proofs in the literature of the fact that every surface can be triangulated are complicated and appeal to geometric intuition. In Section 4 we present a short proof, which is perhaps not easy to follow, but which is simple in the sense that it merely consists of repeated use of the following extension of the Jordan curve theorem: If C_1 and C_2 are simple closed Jordan curves in the plane and f is a homeomorphism between them, then f can be extended to a homeomorphism of the whole plane. This extension, which is called the Jordan-Schönflies theorem is a classical result, which is of interest in its own right. In the present paper it forms a bridge between the Jordan curve theorem and the classification theorem. Although the Jordan-Schönflies theorem may also seem intuitively clear, it does not generalize to sets homeomorphic to a sphere in R^3, as shown by the so-called Alexander's Horned Sphere, see [**5**]. (The Jordan curve theorem does generalize to spheres in R^3.) We present a new (graph-theoretic) proof of the Jordan-Schönflies theorem in Section 3. No previous knowledge of graph theory and only basic topological concepts will be assumed in the paper. In order to emphasize that the proofs are rigorous, no figures (which could be an excuse for lack of details) are included. Instead there are, inevitably, quite a number of technical details in the topological part (Sections 3 and 4). The difficulty in the topological part lies precisely in the details.

The classification of surfaces is not only a beautiful result of considerable independent interest. It has turned out to be a valuable tool in combinatorial analysis. Heawood [**3**] introduced the problem of determining the smallest number $h(S)$ such that every map on the surface S can be coloured in $h(S)$ colours in such a way that no two neighbouring countries receive the same colour. Heawood established an upper bound for $h(S)$. He claimed that his upper bound in fact equals $h(S)$ (except for the sphere) and that this follows by drawing a certain complete graph on S such that no two edges cross. While this claim, which became known as the Heawood conjecture, turned out to be correct, it took almost 80 years before Ringel and Youngs (see [**6**]) completed the proof. One of the main ideas behind the proof is the following: Instead of starting out with S and drawing the complete graph on S, we start out with the complete graph and "paste" discs on it such that we obtain a surface. By the classification theorem and Euler's formula, we know exactly which surface we get, and if we are clever enough, we get S.

The solution of the Heawood problem is an example where the classification theorem plays a role in reducing a problem with a topological content into a purely combinatorial one.

Recently, surfaces have also played a crucial role in a purely combinatorial result with far-reaching consequences in discrete mathematics and theoretical computer science. Let p be a graph property satisfying the following: If G is a graph with property p, then every graph obtained from G by deleting or contracting edges also has property p. The Robertson-Seymour theory [**7**] implies an efficient method (more precisely, a polynomially bounded algorithm) for testing if an arbitrary graph has property p. In particular, for any fixed surface S, there is an efficient algorithm for testing if an arbitrary graph G can be embedded into S, that is, drawn on S such that no two edges cross. In contrast to this, the problem of determining the smallest number of handles that must be added to the sphere in order to get a surface on which G can be embedded is a very difficult one. More precisely, it is NP-complete as shown by the author [**9**].

2. PLANAR GRAPHS AND THE JORDAN CURVE THEOREM. A *simple arc* in a topological space X is the image of a continuous $1-1$ map f from the real interval $[0,1]$ into X. We say that $f(0)$ and $f(1)$ are the *ends* of the arc and that the arc *joins* $f(0)$ and $f(1)$. A *simple closed curve* is defined analogously except that now $f(0) = f(1)$. We say that X is *connected* (more precisely, arcwise connected) if any two elements of X are joined by a simple arc. A *simple polygonal arc* or *closed curve* in the plane is a simple arc or closed curve which is the union of a finite number of straight line segments.

Lemma 2.1. *If Ω is an open connected set in the plane, then any two points in Ω are joined by a simple polygonal arc in Ω.*

Proof: Let p and q be any two points in Ω and let f be a continuous map from $[0, 1]$ to Ω such that $f(0) = p$ and $f(1) = q$. Let A consist of those numbers t in $[0, 1]$ such that Ω contains a simple polygonal arc from p to $f(t)$. Put $t_0 = \sup A$. We must have $t_0 = 1$ since otherwise it is easy to find a t_1 in A such that $t_1 > t_0$, a contradiction. □

A *region* of an open set in the plane is a maximal connected subset. A *graph* G is the union of two finite disjoint sets $V(G)$ and $E(G)$ (called the *vertices* and *edges*, respectively) such that, with every edge, there are associated two distinct vertices x and y, called the *ends* of the edge. We denote such an edge by xy and say that it *joins* x and y or that it is *incident* with x and y. If more than one edge joins x and y we speak of a *multiple* edge. An *isomorphism* between two graphs is defined in the obvious way. A *path* is a graph with distinct vertices $v_1, v_2, \ldots, v_n$ and edges $v_1v_2, v_2v_3, \ldots, v_{n-1}v_n$. If $n \geq 2$ and we add an edge v_nv_1 to this path we obtain a *cycle*. We denote both the above path and cycle by $v_1v_2 \ldots v_n$. (It will always be clear from the context if we are talking about a path or a cycle.) If G is a graph and $A \subseteq V(G) \cup E(G)$, then $G - A$ is the graph obtained from G by deleting all vertices of A and all those edges which are in A or are incident with a vertex in A. We say that G is *connected* if every pair of vertices in G are joined by a path, and G is *2-connected* if it is connected and, for every vertex v, $G - \{v\}$ (which we also denote by $G - v$) is connected. The graph G can be *embedded* in the topological space X if the vertices of G can be represented by distinct elements in X and each edge of G can be represented by a simple arc which joins its two ends in such a way that two edges have at most an end in common. If X is the Euclidean plane R^2, then a graph represented in X is a *plane graph*, and an abstract graph which can be represented in X is a *planar graph*.

Lemma 2.2. *If G is a planar graph, then G can be drawn (embedded) in the plane such that all edges are simple polygonal arcs.*

Proof: Let Γ be a plane graph isomorphic to G. Let p be some vertex of Γ, and let D_p be a closed disc with p as center such that D_p intersects only those edges that are incident with p. Furthermore, assume that $D_p \cap D_q = \emptyset$ for every pair of distinct vertices p, q of Γ. For each edge pq of Γ let C_{pq} be an arc contained in pq such that C_{pq} joins D_p with D_q and has only its ends in common with $D_p \cup D_q$. We can now redraw G such that all arcs C_{pq} are in the new drawing and such that the parts of the edges in the discs D_p are straight line segments. Using Lemma 2.1 it is now easy to replace each C_{pq} by a simple polygonal arc. □

A *subdivision* of a graph G is a graph obtained from G by "inserting vertices on edges." More precisely, some (or all) edges of G are replaced by paths with the same ends. Kuratowski's theorem says that a graph is nonplanar if and only if it contains a subdivision of one of the Kuratowski graph $K_{3,3}$ or K_5. K_5 is the graph on five vertices such that every pair of vertices are joined by exactly one edge. $K_{3,3}$ is the graph with six vertices $v_1, v_2, v_3, u_1, u_2, v_3$ and all nine edges v_iu_j, $1 \leqslant i \leqslant 3$,

$1 \leqslant j \leqslant 3$. A discussion of this fundamental result (including a short proof) can be found in [**8**]. We shall use here only the simple fact that $K_{3,3}$ is nonplanar. For this we need the following special case of the Jordan curve theorem.

Lemma 2.3. *If C is a simple closed polygonal curve in the plane, then $R^2 \setminus C$ has precisely two regions each of which has C as boundary.*

Proof: We first prove that $R^2 \setminus C$ has at most two regions. So suppose (*reductio ad absurdum*) that q_1, q_2, q_3 belong to distinct regions of $R^2 \setminus C$. Select a disc D such that $D \cap C$ is a straight line segment. For each $i = 1, 2, 3$ we can walk along a simple polygonal arc (close to C but not intersecting C) from q_i into D. Hence some two of q_1, q_2, q_3 are connected by a simple polygonal arc, a contradiction.

Next we prove that $R^2 \setminus C$ is not connected. For each point q in $R^2 \setminus C$ we consider a straight half line L starting at q. The intersection $L \cap C$ is a finite number of intervals some of which may be points. Consider such an interval Q. If C enters and leaves Q on the same side of L we will say that C *touches* L at Q. Otherwise C *crosses* L at Q. It is easy to see that the number of times that C crosses L (reduced modulo 2) does not change when the direction of L is changed. So that number depends only on q (and C) and is called the parity of q. Now, the parity is the same for all points on a simple polygonal arc in $R^2 \setminus C$ and hence it is the same for all points in a region of $R^2 \setminus C$. By considering a half line that intersects C precisely once we get points of different parity and hence in different regions. $\square$

The unbounded region of a closed curve C is called the *exterior* of C and is denoted $\text{ext}(C)$. The union of all other regions is the *interior* and is denoted $\text{int}(C)$. Furthermore, we write

$$\overline{\text{int}}(C) = C \cup \text{int}(C) \quad \text{and} \quad \overline{\text{ext}}(C) = C \cup \text{ext}(C).$$

We shall extend Lemma 2.3.

Lemma 2.4. *Let C be a simple closed polygonal curve and P a simple polygonal arc in $\overline{\text{int}}(C)$ such that P joins p and q on C and has no other point in common with C. Let P_1 and P_2 be the two arcs on C from p to q. Then $R^2 \setminus (C \cup P)$ has precisely three regions whose boundaries are C, $P_1 \cup P$, $P_2 \cup P$, respectively.*

Proof: Clearly, $\text{ext}(C)$ is a region of $R^2 \setminus (C \cup P)$. As in the proof of Lemma 2.3 we conclude that the addition of P to C partitions $\text{int}(C)$ into at most two regions. So, we only need to prove that P partitions $\text{int}(C)$ into (at least) two regions. Let L_1, L_2 be crossing line segments such that L_1 is a segment of P, and L_2 has precisely the point in $L_1 \cap L_2$ in common with $C \cup P$. By the proof of Lemma 2.3, the ends of L_2 are in $\text{int}(C)$ and in distinct regions of $R^2 \setminus (P \cup P_1)$, hence also in distinct regions of $R^2 \setminus (P \cup C)$. $\square$

Lemma 2.4 implies that, if r and s are points on $P_1 \setminus \{p, q\}$ and $P_2 \setminus \{p, q\}$, respectively, then it is not possible to join r and s by a simple polygonal arc in $\overline{\text{int}}(C)$ without intersecting P. These remarks also hold when ext and int are interchanged. Hence we get:

Lemma 2.5. *$K_{3,3}$ is nonplanar.*

Proof: $K_{3,3}$ may be thought of as a cycle C: $x_1x_2x_3x_4x_5x_6$ with three chords x_1x_4, x_2x_5, x_3x_6. Now if $K_{3,3}$ were planar we would have a plane drawing such that all edges are simple polygonal arcs, by Lemma 2.2. Then C would be a simple closed polygonal curve and two of the chords x_1x_4, x_2x_5, x_3x_6 would either be in $\text{int}(C)$ or $\text{ext}(C)$. But this would contradict the remark after Lemma 2.4. □

Everything so far is standard and trivial. Now we are ready for the Jordan curve theorem. We remark again that the proof uses only the nonplanarity of $K_{3,3}$.

Proposition 2.6. *If C is a simple closed curve in the plane, then $R^2 \setminus C$ is disconnected.*

Proof: Let L_1 (respectively, L_2) be a vertical straight line intersecting C such that C is entirely in the closed right (respectively, left) half plane of L_1 (respectively, L_2). Let p_i be the top point on $L_i \cap C$ for $i = 1, 2$, and let P_1 and P_2 be the two curves on C from p_1 to p_2. Let L_3 be a vertical straight line between L_1 and L_2. Since $P_1 \cap L_3$ and $P_2 \cap L_3$ are compact and disjoint, L_3 contains an interval L_4 joining P_1 with P_2 and having only its ends in common with C. Let L_5 be a polygonal arc from p_1 to p_2 in $\text{ext}(C)$ consisting of segments of L_1, L_2 and a horizontal straight line segment above C. If L_4 is in $\text{ext}(C)$, then there is a simple polygonal arc L_6 in $\text{ext}(C)$ from L_4 to L_5. But then $C \cup L_4 \cup L_5 \cup L_6$ is a plane graph isomorphic to $K_{3,3}$, contradicting Lemma 2.5. Hence, the midpoint of L_4 does not lie in $\text{ext}(C)$, so $\text{int}(C)$ is nonempty. □

We shall also use the nonplanarity of $K_{3,3}$ to show that $\text{int}(C)$ has only one region. For this we need some graph theoretic facts. First a result on abstract graphs.

Lemma 2.7. *If G is a 2-connected graph and H is a 2-connected subgraph of G, then G can be obtained from H by successively adding paths such that each of these paths joins two distinct vertices in the current graph and has all other vertices outside the current graph.*

Proof: The proof is by induction on the number of edges in $E(G) \setminus E(H)$. If that number is zero, that is, $G = H$, then there is nothing to prove. So assume that $G \neq H$. By the induction hypothesis, Lemma 2.7 holds when the pair G, H is replaced by another pair G', H' such that $E(G') \setminus E(H')$ has fewer edges than $E(G) \setminus E(H)$. Now let H' be a maximal 2-connected proper subgraph of G containing H. If $H' \neq H$ we apply the induction hypothesis to H', H and then to G, H'. So assume that $H' = H$. Since G is connected, there is an edge x_1x_2 in $E(G) \setminus E(H)$ such that x_1 is in H. Since $G - x_1$ is connected, it has a path P: $x_2x_3 \cdots x_k$ such that x_k is in H and all x_i, $2 \leqslant i < k$, are not in H. Possibly $k = 2$. Since $H \cup P \cup \{x_1x_2\}$ is 2-connected, we have $H \cup P \cup \{x_1x_2\} = G$ and the proof is complete. □

If S is a set, then $|S|$ will denote its cardinality.

Lemma 2.8. *If Γ is a plane 2-connected graph with at least three vertices, all of whose edges are simple polygonal arcs, then $R^2 \setminus \Gamma$ has $|E(\Gamma)| - |V(\Gamma)| + 2$ regions each of which has a cycle of Γ as boundary.*

Proof: Let C be a cycle in Γ. By Lemma 2.3, Lemma 2.8 holds if $\Gamma = C$. Otherwise, Γ can be obtained from C by successively adding paths as in Lemma 2.7. Each such path is added in a region. That region is bounded by a cycle and now we apply Lemma 2.4 to complete the proof. (Lemma 2.4 says that the number of regions is increased by 1 when a region is subdivided). $\square$

For a plane graph Γ, the regions of $R^2 \setminus \Gamma$ will also be called *faces* of Γ. The unbounded face is the *outer face* and, if Γ is 2-connected, then the boundary of the outer face is the *outer cycle*.

The union of two abstract graphs is defined in the obvious way. For plane graphs we shall make use of a different type of union.

Lemma 2.9. *If Γ_1 and Γ_2 are two plane graphs such that each edge is a simple polygonal arc, then the union of Γ_1 and Γ_2 is a graph Γ_3.*

Proof: First, let Γ_i' denote the plane graph such that Γ_i' is a subdivision of Γ_i and each edge of Γ_i' is a straight line segment for $i = 1, 2$. Secondly, let Γ_i'' be the subdivision of Γ_i' such that a point p on an edge a of Γ_i' is a vertex of Γ_i'' if either p is a vertex of Γ_{3-i}' or p is on an edge of Γ_{3-i}' that crosses a. Then the usual union of the graphs Γ_1'' and Γ_2'' can play the role of Γ_3. $\square$

If both Γ_1 and Γ_2 in Lemma 2.9 are 2-connected and have at least two points in common, then also Γ_3 is 2-connected.

Lemma 2.10. *Let $\Gamma_1, \Gamma_2, \ldots, \Gamma_k$ be plane 2-connected graphs all of whose edges are simple polygonal arcs such that Γ_i has at least two points in common with each of Γ_{i-1} and Γ_{i+1} and no point in common with any other Γ_j ($i = 2, 3, \ldots, k-1$). Assume also that $\Gamma_1 \cap \Gamma_k = \varnothing$. Then any point which is in the outer face of each of $\Gamma_1 \cup \Gamma_2, \Gamma_2 \cup \Gamma_3 \cdots \Gamma_{k-1} \cup \Gamma_k$ is also in the outer face of $\Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_k$.*

Proof: Suppose p is a point in a bounded face of $\Gamma_1 \cup \cdots \cup \Gamma_k$. Since $\Gamma_1 \cup \cdots \cup \Gamma_k$ is 2-connected, it follows from 2.8 that there is a cycle C in $\Gamma_1 \cup \cdots \cup \Gamma_k$ such that $p \in \mathrm{int}(C)$. Choose C such that C is in $\Gamma_i \cup \Gamma_{i+1} \cup \cdots \cup \Gamma_j$ and such that $j - i$ is minimum. We shall show that $j - i \leqslant 1$. So assume that $j - i \geqslant 2$. Among all cycles in $\Gamma_i \cup \cdots \cup \Gamma_j$ having p in the interior we assume that C is chosen such that the number of edges in C and not in Γ_{j-1} is minimum. Since C intersects both Γ_j and Γ_{j-2}, C has at least two disjoint maximal segments in Γ_{j-1}; let P be one of these; let P' be a shortest path in Γ_{j-1} from P to $C - V(P)$; the ends of P' divide C into arcs P_1 and P_2, each of which contains segments not in Γ_{j-1}. One of the cycles $P' \cup P_1$ and $P' \cup P_2$ contains p in its interior; it has fewer edges not in Γ_{j-1} than C has. This contradicts the minimality of C, so a minimal C does not lie in a minimal union $\Gamma_i \cup \Gamma_{i+1} \cup \cdots \cup \Gamma_j$ with $i \leqslant j - 2$. $\square$

Proposition 2.11. *If P is a simple arc in the plane, then $R^2 \setminus P$ is connected.*

Proof: Let p, q be two points in $R^2 \setminus P$ and let d be a positive number such that each of p, q has distance $> 3d$ from P. We shall join p, q by a simple polygonal arc in $R^2 \setminus P$. Since P is the image of a continuous (and hence uniformly continuous) map we can partition P into segments $P_1, P_2, \ldots, P_k$ such that P_i

joins p_i and p_{i+1} for $i = 1, 2, \ldots, k$ and such that each point on P_i has distance less than d from p_i ($i = 1, 2, \ldots, k-1$). Let d' be the minimum distance between P_i and P_j, $1 \leqslant i \leqslant j - 2 \leqslant k - 2$. Note that $d' \leqslant d$. For each $i = 1, 2, \ldots, k$, we partition P_i into segments $P_{i,1}, P_{i,2}, \ldots, P_{i,k_i}$ such that $P_{i,j}$ joins $p_{i,j}$ with $p_{i,j+1}$ for $j = 1, 2, \ldots, k_i - 1$ and such that each point on $P_{i,j}$ has distance less than $d'/4$ to $p_{i,j}$, and let Γ_i be the graph which is the union of the boundaries of the squares that consist of horizontal and vertical line segments of length $d'/2$ and have a point $p_{i,j}$ as midpoint. Then the graphs $\Gamma_1, \Gamma_2, \ldots, \Gamma_k$ satisfy the assumption of Lemma 2.10. Hence both of p and q are in the outer face of $\Gamma_1 \cup \cdots \cup \Gamma_k$ (because they are outside the disc of radius $3d$ and with center p_i while $\Gamma_i \cup \Gamma_{i+1}$ is inside that disc) and P does not intersect that face. Therefore, p and q can be joined by a simple polygonal arc disjoint from P. □

If C is a closed subset of the plane and Ω is a region of $R^2 \setminus C$, then a point p in C is *accessible* from Ω if for some (and hence each) point q in Ω, there is a simple polygonal arc from q to p having only p in common with C. If C is a simple closed curve, then p need not be accessible from Ω. However, if P is any arc of C containing p, then Proposition 2.11 implies that $R^2 \setminus (C \setminus P)$ is connected and therefore contains a simple polygonal arc P' from q to a region of $R^2 \setminus C$ distinct from Ω. Then P' intersects C in a point on P. Since P can be chosen to be arbitrarily small we conclude that the points on C accessible from Ω are dense on C. We also get

Theorem 2.12 (*The Jordan Curve Theorem*). *If C is a simple closed curve in the plane, then $R^2 \setminus C$ has precisely two regions, each of which has C as boundary.*

Proof: Assume (*reductio ad absurdum*) that q_1, q_2, q_3 are points in distinct regions $\Omega_1, \Omega_2, \Omega_3$ of $R^2 \setminus C$. Let Q_1, Q_2, Q_3 be pairwise disjoint segments of C. By the remark following Proposition 2.11, Ω_i has a simple polygonal arc $P_{i,j}$ from q_i to Q_j for $i, j = 1, 2, 3$. We can assume that $P_{i,j} \cap P_{i,j'} = \{q_i\}$ for $j \neq j'$. (If we walk along $P_{i,2}$ from Q_2 towards q_i and we hit $P_{i,1}$ in $q_i' \neq q_i$, then we can modify $P_{i,2}$ such that its last segment is close to the segment of $P_{i,1}$ from q_i' to q_i and such that the new $P_{i,2}$ has only q_i in common with $P_{i,1}$. $P_{i,3}$ can be modified similarly, if necessary.) Clearly, $P_{i,j} \cap P_{i',j'} = \varnothing$ when $i \neq i'$. We can now extend (by adding a segment in each of Q_1, Q_2, Q_3) the union of the arcs $P_{i,j}$ ($i, j = 1, 2, 3$) to a plane graph isomorphic to $K_{3,3}$. This contradicts Lemma 2.5. Thus $R^2 \setminus C$ has precisely two regions $\text{ext}(C)$ and $\text{int}(C)$. As above, Proposition 2.11 implies that every point of C is a boundary point of $\text{ext}(C)$ and $\text{int}(C)$. □

The Jordan Curve Theorem is a special case of the Jordan-Schönflies theorem which we prove in the next section. For this we shall generalize some of the previous results. First, Lemma 2.4 generalizes as follows.

Lemma 2.13. *Let C be a simple closed curve and P a simple polygonal arc in* $\text{int}(C)$ *such that P joins p and q on C and has no other point in common with C. Let P_1 and P_2 be the two arcs on C from p to q. Then $R^2 \setminus (C \cup P)$ has precisely three regions whose boundaries are C, $P_1 \cup P$, and $P_2 \cup P$, respectively.*

Proof: As in the proof of Lemma 2.4 the only nontrivial part is to prove that $\overline{\text{int}}(C)$ is partitioned into (at least) two regions. If the ends of L_2 (defined as in the proof of Lemma 2.4) are in the same region of $R^2 \setminus (P \cup C)$, then that region contains a polygonal arc P_3 such that $P_3 \cup L_2$ is a simple closed polygonal curve. By the proof of Lemma 2.3, the ends of L_1 are in distinct regions of $R^2 \setminus (P_3 \cup L_2)$. But they are also in the same region of $R^2 \setminus (P_3 \cup L_2)$ since they are joined by a simple arc (in $P \cup C$) not intersecting $P_3 \cup L_2$. This contradiction proves Lemma 2.13. □

We also generalize Lemma 2.8.

Lemma 2.14. *If* Γ *is a plane 2-connected graph containing a cycle* C *(which is a simple closed curve) such that all edges in* $\Gamma \setminus C$ *are simple polygonal arcs in* $\overline{\text{int}}(C)$, *then* $R^2 \setminus \Gamma$ *has* $|E(\Gamma)| - |V(\Gamma)| + 2$ *regions each of which has a cycle of* Γ *as boundary.*

Proof: The proof is as that of Lemma 2.8 except that we now use Lemma 2.13 instead of Lemma 2.4. □

Finally, we shall use the fact that Lemma 2.9 remains valid if Γ_1 and Γ_2 are plane graphs whose intersection contains a cycle C such that all edges in Γ_1 or Γ_2 (not in C) are simple polygonal arcs in $\overline{\text{int}}(C)$.

3. THE JORDAN-SCHÖNFLIES THEOREM. If C and C' are simple closed curves and Γ and Γ' are 2-connected graphs consisting of C (respectively, C') and simple polygonal arcs in $\overline{\text{int}}(C)$ (respectively, $\overline{\text{int}}(C')$), then Γ and Γ' are said to be *plane-isomorphic* if there is an isomorphism of Γ to Γ' such that a cycle in Γ is a face boundary of Γ iff the image of the cycle is a face boundary of Γ' and such that the outer cycle of Γ is mapped onto the outer cycle of Γ'.

Theorem 3.1. *If* f *is a homeomorphism of a simple closed curve* C *onto a simple closed curve* C', *then* f *can be extended into a homeomorphism of the whole plane.*

Proof: Without loss of generality we can assume that C' is a convex polygon. We shall first extend f to a homeomorphism of $\overline{\text{int}}(C)$ to $\overline{\text{int}}(C')$. Let B denote a countable dense set in $\text{int}(C)$ (for example the points with rational coordinates). Since the points on C accessible from $\text{int}(C)$ are dense on C, there exists a countable dense set A in C consisting of points accessible from $\text{int}(C)$. Let $p_1, p_2, \dots$ be a sequence of points in $A \cup B$ such that each point in $A \cup B$ occurs infinitely often in that sequence. Let Γ_0 denote any 2-connected graph consisting of C and some simple polygonal arcs in $\overline{\text{int}}(C)$. Let Γ'_0 be a graph consisting of C' and simple polygonal arcs in $\overline{\text{int}}(C')$ such that Γ_0 and Γ'_0 are plane-isomorphic (with isomorphism g_0) such that g_0 and f coincide on $C \cap V(\Gamma_0)$. We now extend f to $C \cup V(\Gamma_0)$ such that g_0 and f coincide on $V(\Gamma_0)$. We shall define a sequence of 2-connected graphs $\Gamma_0, \Gamma_1, \Gamma_2 \dots$ and $\Gamma'_0, \Gamma'_1 \dots$ such that, for each $n \geq 1$, Γ_n is an extension of a subdivision of Γ_{n-1}, Γ'_n is an extension of a subdivision of Γ'_{n-1}, there is a plane isomorphism g_n of Γ_n onto Γ'_n coinciding with g_{n-1} on $V(\Gamma_{n-1})$, and Γ_n (respectively Γ'_n) consists of C (respectively C') and simple polygonal arcs in $\overline{\text{int}}(C)$ (respectively $\overline{\text{int}}(C')$). Also, we shall assume that $\Gamma'_n \setminus C'$ is connected for each n. We then extend f to $C \cup V(\Gamma_n)$ such that f and g_n coincide on $V(\Gamma_n)$.

Suppose we have already defined $\Gamma_0, \Gamma_1, \dots, \Gamma_{n-1}, \Gamma'_0, \Gamma'_1, \dots, \Gamma'_{n-1}$, and $g_0, g_1, \dots, g_{n-1}$. We shall define Γ_n, Γ'_n and g_n as follows. We consider the point p_n. If $p_n \in A$, then we let P be a simple polygonal arc from p_n to a point q_n of $\Gamma_{n-1} \setminus C$ such that $\Gamma_{n-1} \cap P = \{p_n, q_n\}$. We let Γ_n denote the graph $\Gamma_{n-1} \cup P$. P

is drawn in a face of Γ_{n-1} bounded by a cycle S, say. We add to Γ'_{n-1} a simple polygonal arc P' in the face bounded by $g_{n-1}(S)$ such that P' joins $f(p_n)$ with $g_{n-1}(q_n)$ (if q_n is a vertex of Γ_{n-1}) or a point on $g_{n-1}(a)$ (if a is an edge of Γ_{n-1} containing the point q_n). Then we put $\Gamma'_n = \Gamma'_{n-1} \cup P'$ and we define the plane-isomorphism g_n from Γ_n to Γ'_n in the obvious way. We extend f such that $f(q_n) = g_n(q_n)$.

If $p_n \in B$ we consider the largest square which has vertical and horizontal sides, which has p_n as midpoint and which is in $\overline{\text{int}}(C)$. In this square (whose sides we are not going to add to Γ_{n-1} as they may contain infinitely many points of C) we draw a new square with vertical and horizontal sides each of which has distance $< 1/n$ from the sides of the first square. Inside the new square we draw vertical and horizontal lines such that p_n is on both a vertical line and a horizontal line and such that all regions in the square have diameter $< 1/n$. We let H_n be the union of Γ_{n-1} and the new horizontal and vertical straight line segments possibly together with an additional polygonal arc in int(C) in order to make H_n 2-connected and $H_n \setminus C$ connected. By Lemma 2.7, H_n can be obtained from Γ_{n-1} by successively adding paths in faces. We add the corresponding paths to Γ'_{n-1} and obtain a graph H'_n which is plane-isomorphic to H_n. Then we add vertical and horizontal lines in $\overline{\text{int}}(C')$ to H'_n such that the resulting graph has no (bounded) region of diameter $\geqslant 1/2n$. If necessary, we displace some of the lines a little such that they intersect C' only in $f(A)$ and such that all bounded regions have diameter $< 1/n$ and such that each of the new lines has only finite intersection with H'_n. This extends H'_n into a graph we denote by Γ'_n. We add to H_n polygonal arcs such that we obtain a graph Γ_n plane-isomorphic to Γ'_n. Then we extend f such that it is defined on $C \cup V(\Gamma_n)$ and coincides with the plane-isomorphism g_n on $V(\Gamma_n)$. When we extend H'_n into Γ'_n and H_n into Γ_n we are adding many edges and it is perhaps difficult to visualize what is going on. However, Lemma 2.7 tells us that we can look at the extension of H'_n into Γ'_n as the result of a sequence of simple extensions each consisting of the addition of a path (which in this case is a straight line segment in a face). We then just perform successively the corresponding additions in H_n. Note that we have plenty of freedom for that since the current f is only defined on the current vertex set. The images of the points on the current edges have not been specified yet. In this way we extend f to a $1-1$ map defined on $F = C \cup V(\Gamma_0) \cup V(\Gamma_1) \cup \cdots$ and with image $C' \cup V(\Gamma'_0) \cup V(\Gamma'_1) \cup \cdots$. These sets are dense in $\overline{\text{int}}(C)$ and $\overline{\text{int}}(C')$, respectively. If p is a point in int(C) on which f is not yet defined, then we consider a sequence $q_1, q_2, \dots$ converging to p and consisting of points in $V(\Gamma_0) \cup V(\Gamma_1) \cup \cdots$. We shall show that $f(q_1), f(q_2), \dots$ converges and we let $f(p)$ be the limit. Let d be the distance from p to C and let p_n be a point of B of distance $< d/3$ from p. Then p is inside the largest square in $\overline{\text{int}}(C)$ having p_n as midpoint (and also inside what we called the new square if n is sufficiently large). By the construction of Γ_n and Γ'_n it follows that Γ_n has a cycle S such that $p \in \text{int}(S)$ and such that both S and $g_n(S)$ are in discs of radius $< 1/n$. Since f maps $F \cap \text{int}(S)$ into $\text{int}(g_n(S))$ and $F \cap \overline{\text{ext}}(S)$ into $\overline{\text{ext}}(g_n(S))$, it follows in particular, that the sequence $f(q_m), f(q_{m+1}), \dots$ is in $\text{int}(g_n(S))$ for some m. Since n can be chosen arbitrarily large, $f(q_1), f(q_2), \dots$ is a Cauchy sequence and hence convergent. It follows that f is well-defined. Moreover, using the above notation, f maps int(S) into $\text{int}(g_n(S))$. Hence f is continuous in int(C). Since $V(\Gamma'_0) \cup V(\Gamma'_1)$ is dense in int(C') the same argument shows that f maps int(C) onto int(C') that f is $1-1$ and that f^{-1} is continuous on int(C'). It only remains to be shown that f is continuous on C. (Then also f^{-1} is continuous since $\overline{\text{int}}(C)$ is compact). In order to prove this it is sufficient to consider a sequence $q_1, q_2, \ldots$ of points in int(C) converging to q on C and then show that $f(q_1), f(q_2), \ldots$ converges to $f(q)$. Suppose therefore that this is not the case. Since $\overline{\text{int}}(C')$ is compact we can assume (by considering an appropriate subsequence, if necessary) that $\lim_{n \to \infty} f(q_n) = q' \neq f(q)$. Since f^{-1} is continuous on int(C'), q' is on C'. Since A is dense in C, $f(A)$ is dense in C' and hence each of the two arcs on C' from q' to $f(q)$ contain a point $f(q_1)$ and $f(q_2)$, respectively, in $f(A)$. For some n, Γ_n has a path P from q_1 to q_2 having only q_1 and q_2 in common with C. By Lemma 2.13, P separates int(C) into two regions. These two regions are mapped on the two distinct regions of $\text{int}(C') \setminus g_n(P)$. One of these contains almost all the $f(q_n)$ while the other has $f(q)$ on its boundary, but not the boundary common to both regions. Hence we cannot have $\lim_{n \to \infty} f(q_n) = q'$. This contradiction shows that f has the appropriate extension to int(C).

By similar arguments, f can be extended to ext(C). We consider a coordinate system in the plane. Without loss of generality we can assume that int(C) contains the origin and that both C and C' are in the interior of the quadrangle T with corners $(\pm 1, \pm 1)$. Let L_1, L_2, L_3 be the line segments (on lines through the origin) from $(1, 1)$, $(-1, -1)$ and $(1, -1)$, respectively, to C. Let p_i be the end of L_i on C for $i = 1, 2, 3$. Let L'_1 and L'_2 be polygonal arcs from $f(p_1)$ to $(1, 1)$ and from $f(p_2)$ to $(-1, -1)$, respectively, such that $L'_1 \cap L'_2 = \varnothing$ and L'_i has only its ends in common with C' and T for $i = 1, 2$. It is easy to see that we can find a polygonal arc L'_3 from $f(p_3)$ to either $(1, -1)$ or $(-1, 1)$ such that L'_3 is disjoint from $L'_1 \cup L'_2$ and has only its ends in common with C' and T. After a reflection of C' in the line through $(1, 1)$ and $(-1, -1)$, if necessary, we can assume that L'_3 goes to $(1, -1)$. Now we can use the method of the first part of the proof to extend f to a homeomorphism of $\overline{\text{int}}(T)$ such that f is the identity on T. Then f extends to a homeomorphism of the whole plane such that f is the identity on ext(T). □

If F is a closed set in the plane, then we say that point p in F is *curve-accessible* if, for each point q not in F, there is a simple arc from q to p having only p in common with F. The Jordan-Schönflies theorem implies that every point on a simple closed curve is curve-accessible. Hence we have the following extension of part of Theorem 2.12.

Theorem 3.2. *If F is a closed set in the plane with at least three curve-accessible points, then $R^2 \setminus F$ has at most two regions.*

Proof: If p_1, p_2, p_3 are curve-accessible on F and q_1, q_2, q_3 belong to distinct regions of $R^2 \setminus F$, then we get, as in the proof of Theorem 2.12, a plane graph isomorphic to $K_{3,3}$ with vertices $p_1, p_2, p_3, q_1, q_2, q_3$, a contradiction to Lemma 2.5. □

In Theorem 3.2, "three" cannot be replaced by "two." To see this, we let F be a collection of internally disjoint simple arcs between two fixed points.

Theorem 3.3. *Let Γ and Γ' be 2-connected plane graphs such that g is a homeomorphism and plane-isomorphism of Γ onto Γ'. Then g can be extended to a homeomorphism of the whole plane.*

Proof: The proof is by induction on the number of edges of Γ. If Γ is a cycle, then Theorem 3.3 reduces to Theorem 3.1. Otherwise it follows from Lemma 2.7 that Γ has a path P and a 2-connected subgraph Γ_1 containing the outer cycle of Γ

such that Γ is obtained from Γ_1 by adding P in $\overline{\text{int}}(C)$ where C bounds a face of Γ_1. Now we apply the induction hypothesis first to Γ_1 and then to the two cycles of $C \cup P$ containing P.

4. TRIANGULATING A SURFACE. Consider a finite collection of pairwise disjoint convex polygons (together with their interiors) in the plane such that all side lengths are 1. Form a topological space S as follows: Every side in a polygon is identified with precisely one side in another (or in the same) polygon. This also defines a graph G whose vertices are the corners and the edges the sides. Clearly S is compact. Now S is a surface iff S is connected (i.e., G is connected) and S is locally homeomorphic to a disc at every vertex v of G. If this is the case then we say that G is a *2-cell embedding* in S. If all polygons are triangles, then we say that G is a *triangulation* of S and that S is a *triangulated surface*. In case of a triangulation we shall assume that there are at least four triangles and that there are no multiple edges.

Theorem 4.1. *Every surface S is homeomorphic to a triangulated surface.*

Proof: Since the interior of a convex polygon can be triangulated it is sufficient to prove that S is homeomorphic to a surface with a 2-cell embedding. For each point p on S, let $D(p)$ be a disc in the plane which is homeomorphic to a neighbourhood of p on S. (Instead of specifying a homeomorphism we shall use the same notation for a point in $D(p)$ and the corresponding point on S.) In $D(p)$ we draw two quadrangles $Q_1(p)$ and $Q_2(p)$ such that $p \in \text{int}(Q_1(p)) \subset \text{int}(Q_2(p))$. Since S is compact, it has a finite number of points $p_1, p_2, \dots, p_n$ such that $S = \bigcup_{i=1}^{n} \text{int}(Q_1(p_i))$. Viewed as subsets in the plane, $D(p_1), \dots, D(p_n)$ can be assumed to be pairwise disjoint. In what follows we are going to keep $D(p_1), D(p_2), \dots, D(p_n)$ fixed in the plane (keeping in mind, though, that they also correspond to subsets of S). However, we shall modify the homeomorphism between $D(p_i)$ and the corresponding set on S and consider new quadrangles $Q_1(p_i)$. More precisely, we shall show that $Q_1(p_1), \dots, Q_1(p_n)$ can be chosen such that they form a 2-cell embedding of S.

Suppose, by induction on k, that they have been chosen such that any two of $Q_1(p_1), Q_1(p_2), \dots, Q_1(p_{k-1})$ have only a finite number of points in common on S. We now focus on $Q_2(p_k)$. We define a *bad segment* as a segment P of some $Q_1(p_j)$ $(1 \leqslant j \leqslant k-1)$ which joins two points of $Q_2(p_k)$ and which has all other points in $\text{int}(Q_2(p_k))$. Let $Q_3(p_k)$ be a square between $Q_1(p_k)$ and $Q_2(p_k)$. We say that a bad segment inside $Q_2(p_k)$ is *very bad* if it intersects $Q_3(p_k)$. There may be infinitely many bad segments but only finitely many very bad ones. The very bad ones together with $Q_2(p_k)$ form a 2-connected graph Γ. We redraw Γ inside $Q_2(p_k)$ such that we get a graph Γ' which is plane-isomorphic to Γ and such that all edges of Γ' are simple polygonal arcs. This can be done using Lemma 2.7. Now we apply Theorem 3.3 to extend the plane-isomorphism from Γ to Γ' to a homeomorphism of $\overline{\text{int}}\, Q_2(p_k)$ keeping $Q_2(p_k)$ fixed. This transforms $Q_1(p_k)$ and $Q_3(p_k)$ into simple closed curves Q_1' and Q_3' such that $p_k \in \text{int}\, Q_1' \subseteq \text{int}\, Q_3'$. We consider a simple closed polygonal curve Q_3'' in $\text{int}\, Q_2(p_k)$ such that $Q_1' \subseteq \text{int}\, Q_3''$ and such that Q_3'' intersects no bad segments except the very bad ones (which are now simple polygonal arcs). (The existence of Q_3'' can be established as follows: For every point p on Q_3' we let $R(p)$ be a square with p as midpoint such that $R(p)$ does not intersect either Q_1' nor any bad segment which is not very bad. We consider a (minimal) finite covering of Q_3' by such squares. The union of those squares is a 2-connected plane graph whose outer cycle can play the role of Q_3''). By redrawing $\Gamma' \cup Q_3''$ (which is a 2-connected graph) and using Theorem 3.3 once more we can assume that Q_3'' is in fact a quadrangle having Q_1' in its interior. If we let Q_3'' be the new choice of $Q_1(p_k)$, then any two of $Q_1(p_1), \dots, Q_1(p_k)$ have only finite intersection. The inductive hypothesis is proved for all k.

Thus we can assume that there are only finitely many very bad segments inside each $Q_2(p_k)$ and that those segments are simple polygonal arcs forming a 2-connected plane graph. The union $\bigcup_{i=1}^{n} Q_1(p_i)$ may be thought of as a graph Γ drawn on S. Each region of $S \setminus \Gamma$ is bounded by a cycle C in Γ. (We may think of C as a simple closed polygonal curve inside some $Q_2(p_i)$). Now we draw a convex polygon C' of side length 1 such that the corners of C' correspond to the vertices of C. The union of the polygons C' forms a surface S' with a 2-cell embedding Γ' which is isomorphic to Γ. An isomorphism of Γ to Γ' may be extended to a homeomorphism f of the point set of Γ on S onto the point set of Γ' on S'. In particular, the restriction of f to the above cycle C is a homeomorphism onto C'. By Theorem 3.1, f can be extended to a homeomorphism of $\overline{\text{int}}(C)$ to $\overline{\text{int}}(C')$. This defines a homeomorphism of S onto S'. □

5. THE CLASSIFICATION OF SURFACES. Consider now two disjoint triangles T_1, T_2 (such that all six sides have the same length) in a face F of a surface S with a 2-cell embedding G. We form a new surface S' by deleting from F the interior of T_1 and T_2 and identifying T_1 with T_2 such that the clockwise orientations around T_1 and T_2 disagree. (We recall that S consists of polygons and their interiors in the plane. So when we speak of clockwise orientation we are simply referring to the plane. We are not discussing orientability of surfaces.) If the orientations agree we obtain instead a surface S''. Finally, we let S''' denote the surface obtained by deleting the interior of T_1 and identifying "diametrically opposite" points on T_1. We say that S', S'', S''' are obtained from S by adding a *handle*, a *twisted handle*, and a *crosscap*, respectively. It is easy to extend G to a 2-cell embedding of S', S'' and S''', respectively. Also, it is an easy exercise to show that S', S'' and S''' are independent (up to homeomorphism) of where T_1 and T_2 are located since it is easy to continuously deform a pair of triangles into another pair of triangles inside a given triangle. In fact, they may belong to distinct faces, also, except that then (at this stage) we cannot distinguish between a handle and a twisted handle. When adding a crosscap it is sufficient that T_1 is a simple closed polygonal curve, which can be continuously deformed into a point (and hence to a triangle in a face).

We shall now consider all surfaces obtained from the sphere S_0 (which we here think of as a tetrahedron) by adding handles, twisted handles and crosscaps. If we add to S_0 h handles we obtain the surface S_h, and if we add to S_0 k crosscaps we obtain N_k. S_1, N_1, N_2 are the *torus*, the *projective plane* and the *Klein bottle*, respectively. N_2 is also S_0 plus a twisted handle. One way to see this is as follows: Let T_1 and T_2 be two disjoint tetrahedra (which are homeomorphic to S_0). Select a triangle in T_1 and T_2 and add in that triangle a twisted handle or two crosscaps. This transforms T_1 into T_1' and T_2 into T_2'. Now choose your favourite representation of the Klein bottle and your favourite triangulation G of it. Then for each $i = 1, 2$, draw G on T_i' such that the face boundaries are the same triangles in G in all three triangulations. Then the graph isomorphism of G on T_1' to G on T_2' can be extended to a homeomorphism of T_1' onto T_2'. Moreover, if we have already added a crosscap, then adding a handle amounts to the same, up to homeomorphism, as adding a twisted handle. (First observe that when we add a crosscap, it does not matter where we add it; we get always the same surface up to homeomorphism. So we only need to verify the statement when we add a crosscap and then a handle or twisted handle inside the same triangle of the surface. This can be done by triangulating the two surfaces by the same graph G as above). So, the surfaces obtained from S_0 by adding handles, twisted handles and crosscaps are precisely the surfaces S_h $(h \geq 0)$ and N_k $(k \geq 1)$.

Theorem 5.1. *Let S be a surface and G a 2-cell embedding of S with n vertices, e edges and f faces. Then S is homeomorphic to either S_h or N_k where h and k are defined by the equations*

$$n - e + f = 2 - 2h = 2 - k.$$

Proof: We first show that $n - e + f \leqslant 2$. For this we successively delete edges from G until we get a minimal connected subgraph of G, that is, a spanning tree H of G. For each edge deletion the number of faces (which are now not necessarily 2-cells) is unchanged or decreased by 1. Since H has n vertices, $n - 1$ edges and only one face it follows that $n - e + f \leqslant 2$.

We next extend G to a triangulation of S as follows: For each face F of G which is a convex polygon with corners $v_1, v_2, \dots, v_q$, where $q \geqslant 4$ and their indices are expressed modulo q, we add new vertices $u, u_1, \dots, u_q$ in F and we add the edges $u_i v_i, u_i v_{i+1}, u_i u_{i+1}, u_i u$ for $i = 1, 2, \dots, q$. Let n', e', f' be the number of vertices edges and faces, respectively, of G'. Clearly, $n' - e' + f' = n - e + f$. Thus it is sufficient to prove the Theorem in the case where G is a triangulation which we now assume. Suppose (*reductio ad absurdum*) that S, G are a counterexample to Theorem 5.1 such that G is a triangulation with at least four vertices and

(1) $2 - n + e - f$ is minimum.

(2) n is minimum subject to (1), and

(3) the minimum valency m of G is minimum subject to (1), (2). (The *valency* of a vertex is the number of edges incident with it.)

Let v be a vertex of minimum valency. Let $v_1, v_2, \dots, v_m$ be the neighbours of v such that $vv_1v_2, vv_2v_3, \dots, vv_mv_1$ are the faces incident with v and the indices are expressed modulo m. Since v_1 and v_m are joined only by one edge, $m \geq 3$. If $m = 3$, then $G - v$ is a triangulation of S unless $n = 4$ in which case S is the tetrahedron. This contradicts (2) or the assumption that S, G are a counterexample to the Theorem. So $m \geqslant 4$.

If for some $i = 1, 2, \dots, m$, v_i is not joined to v_{i+2} by an edge, then we let G' be obtained from G by deleting the edge vv_{i+1} and adding the edge $v_i v_{i+2}$ instead. Clearly, G' triangulates S, contradicting (3). So we can assume that G contains all edges $v_i v_{i+2}$, $i = 1, 2, \dots, m$, when v is a vertex of minimum valency.

Intuitively, we complete the proof by cutting the surface (using a pair of scissors, say) along the triangle T: vv_1v_3. This transforms T into either two triangles T_1 and T_2 or into a hexagon H (in case S has a Möbius strip that contains T). We get a new surface S' by adding two new triangles (and their interior) or a hexagon (and its interior which we triangulate) and identify their sides with T_1 and T_2 or with H, respectively. Then S' is a triangulated surface with smaller $2 - n + e - f$ than S. By the minimality of this parameter, S' is of the form S_g, or N_k. Then S is of that form, too.

Formally, we argue as follows. Recall that S is a triangulated surface, i.e., S is obtained by identifying sides of pairwise disjoint triangles in the plane. Let M denote the topological space which is formed by using the same triangles and the same side identifications, except that those six sides that correspond to the edges vv_1, v_1v_3, v_3v are not identified with any other side. Let us call those six sides *boundary sides* of M. Let G' be the graph whose vertices are the corners of the triangles of M and whose edges are the sides of the triangles. It is easy to see that G' has precisely six vertices which are incident with boundary sides and that each of these six vertices is incident with precisely two boundary sides. Thus the boundary sides are a subgraph C of G' with vertices each of which has valency 2. There are only two such graphs (up to isomorphism): C is either a hexagon or two disjoint triangles. If C is two disjoint triangles, then we add to M two disjoint triangles (and their interior) in the plane and identify their sides with the edges of C such that we obtain a new surface S' which is triangulated by G'. If C is a hexagon, then we add to M a hexagon in the plane together with its interior (which we triangulate) and then we identify the sides of this hexagon with the edges of C. In this way M is extended to a surface S'' and G' is extended to a graph G'' which triangulates S''. Thus we have transformed G and S into a triangulation G' with n' vertices e' edges and f' faces of a surface S', or a triangulation G'' with n'' vertices e'' edges and f'' faces of a surface S''. In the former case we have

$$e' - n' + f' = e - n + f + 2.$$

In the latter case we have

$$e'' - n'' + f'' = e - n + f + 1.$$

By (1), S' or S'' is homeomorphic to a surface of the form $S_{h'}$ or $N_{k'}$. (Note that G' is obtained from G by "cutting" the triangle vv_1v_3. Then G' is connected because of the edge v_2v_m. Hence also the spaces M, S', S'' are connected.) If C consists of two triangles, then clearly S is obtained from S' by adding a handle or a twisted handle. If C is a hexagon, then in S'', C can be continuously deformed into a point, and hence S is obtained from S'' by adding a crosscap (see the discussion preceding Theorem 5.1). In the latter case (where C is a hexagon) S is homeomorphic to $N_{k'+1}$ or $N_{2h'+1}$ (by the discussion preceding Theorem 5.1). This contradicts the assumption that S and G are a counterexample to Theorem 5.1. Similarly, if C is two triangles, then S is homeomorphic to either $N_{k'+2}$ or $S_{h'+1}$ or $N_{2h'+2}$, and again we obtain a contradiction which finally proves the theorem. □

We have now completed the proof of the classification theorem without referring to orientability of surfaces or using Euler's formula (which consists of the equations of Theorem 5.1 and which is therefore now a corollary of Theorem 5.1). To complete the discussion we indicate a proof of the fact that all the surfaces $S_0, S_1, \ldots, N_1, N_2 \ldots$ are pairwise nonhomeomorphic. In this discussion, however, many details will be left for the reader.

First we observe that Euler's formula holds for all 2-cell embeddings since any such embedding can be extended to a triangulation. Now let us consider any connected graph G with n vertices and e edges drawn on S_h. Using Lemma 2.2 we assume that all edges are simple polygonal arcs. Let f be the number of faces for this drawing. If G' is a 2-cell embedding of S_h, then $G \sqcup G'$ is a 2-cell embedding satisfying Euler's formula and containing a subdivision of G. By successively deleting edges (and isolated vertices) from $G \sqcup G'$ until we get a subdivision of G we conclude that

$$n - e + f \geqslant 2 - 2h.$$

Since

$$3f \leqslant 2e$$

we conclude that

$$e \leqslant 3n - 6 + 6h$$

with equality if and only if G is a triangulation of S_h. Thus a triangulation of S_h has too many edges in order to be drawn on $S_{h'}$ when $h' < h$, and hence S_h and $S_{h'}$ are nonhomeomorphic for $h' < h$. More generally, this argument shows that $S_0, S_1, \ldots, N_1, N_2, \ldots$ are pairwise nonhomeomorphic except that S_h and N_{2h} might be homeomorphic. We sketch an argument which shows that they are not.

It is easy to describe a simple closed polygonal curve C in N_{2h} such that, when we traverse C, left and right interchange. Also it is easy (though a little tedious) to show that S_h has no such simple closed polygonal curve C'. (It is convenient to consider a 2-cell embedding G such that G contains no such C' and then extend the argument to an arbitrary C' in S.) So it suffices to show the following: If there exists a homeomorphism f: $N_{2h} \to S_h$, then there exists a homeomorphism f': $N_{2h} \to S_h$ such that $f'(C)$ is a simple closed polygonal curve. To see this we let G be a 2-cell embedding of N_{2h}. Then also $G \sqcup C$ may be regarded as a 2-cell embedding, and $G \sqcup C$ can be extended to a triangulation H of N_{2h}. We construct H such that it has no other triangles than the face boundaries. Then $\varphi(H)$ is a graph drawn on S_h and we apply Lemma 2.2 to redraw $\varphi(H)$ (resulting in a graph H') such that all edges are simple polygonal arcs. Since H' and H are isomorphic and H is a triangulation of N_{2h}, it follows from Euler's formula that H' is a triangulation of S_h. Hence the face boundaries of H' are the same as the face boundaries of H. So, any isomorphism $H \to H'$ can be extended into a homeomorphism φ': $N_{2h} \to S_h$ taking C into a simple closed polygonal curve.

ACKNOWLEDGMENT. Thanks are due to the referee for numerous comments on the paper.

REFERENCES

1. P. Andrews, The classification of surfaces, *Amer. Math. Monthly*, to appear.
2. J. L. Gross and T. W. Tucker, *Topological Graph Theory*, Wiley and Sons, New York, 1987.
3. P. J. Heawood, Map-color theorem, *Quart. J. Math. Oxford Ser.*, 24 (1890) 332–338.
4. B. Kerékjártó, *Vorlesungen über Topologie*, Springer, Berlin, 1923.
5. E. E. Moise, *Geometric Topology in Dimensions 2 and 3*, Graduate Texts in Mathematics, Springer, New York 1977.
6. G. Ringel, *Map Color Theorem*, Springer-Verlag, Berlin, 1974.
7. N. Robertson and P. D. Seymour, Graph minors XIII. The disjoint paths problem, to appear.
8. C. Thomassen, Kuratowski's theorem, *J. Graph Theory*, 5 (1981) 225–241.
9. C. Thomassen, The graph genus problem is NP-complete, *J. Algorithms*, 10 (1989) 568–576.
10. C. Thomassen, Embeddings and minors, in: *Handbook of Combinatorics* (eds., M. Grötschel, L. Lovász and R. L. Graham), North-Holland, to appear.
11. W. T. Tutte, *Graph Theory*, Addison-Wesley, Reading, Mass., 1984.
12. H. Tverberg, A proof of the Jordan Curve Theorem, *Bull. London Math. Soc.* 12 (1980) 34–38.

Mathematical Institute
The Technical University of Denmark
Building 303
DK-2800 Lyngby, DENMARK

Are Mathematics and Poetry Fundamentally Similar?

JoAnne S. Growney

If you doubt their intrinsic similarity, consider the following quotations. In each of the following, the key word ("mathematics" or "poetry" or "mathematician" or "poet" or a variation of one of these terms) has been left out, although the name of the author may provide a give-away clue. Can you guess which art form is being described in each case? The missing words are supplied at the end of the quotations.

(1) ________ is the art of uniting pleasure with truth. —*Samuel Johnson*

(2) To think is thinkable—that is the ________'s aim. —*Cassius J. Keyser*

(3) All ________ [is] putting the infinite within the finite. —*Robert Browning*

(4) The moving power of ________ invention is not reasoning but imagination. —*A. DeMorgan*

(5) When you read and understand ________, comprehending its reach and formal meanings, then you master chaos a little. —*Stephen Spender*

(6) ________ practice absolute freedom. —*Henry Adams*

(7) I think that one possible definition of our modern culture is that it is one in which nine-tenths of our intellectuals can't read any ________. —*Randall Jarrell*

(8) Do not imagine that ________ is hard and crabbed, and repulsive to common sense. It is merely the etherealization of common sense. —*Lord Kelvin*

(9) The merit of ________, in its wildest forms, still consists in its truth; truth conveyed to the understanding, not directly by words, but circuitously by means of imaginative associations, which serve as conductors. —*T. B. Macaulay*

(10) It is a safe rule to apply that, when a ________ or philosophical author writes with a misty profundity, he is talking nonsense. —*A. N. Whitehead*

(11) ________ is a habit. —*C. Day-Lewis*

(12) ... in ________ you don't understand things, you just get used to them. —*John von Neumann*

(13) ________ are all who love—who feel great truths
And tell them. —*P. J. Bailey*
Festus

(14) The ________ is perfect only in so far as he is a perfect being, in so far as he perceives the beauty of truth; only then will his work be thorough, transparent, comprehensive, pure, clear, attractive, and even elegant. —*Goethe*

(15) ... [In these days] the function of ________ as a game ... [looms] larger than its function as a search for truth .... —*C. Day-Lewis*

(16) A thorough advocate in a just cause, a penetrating ________ facing the starry heavens, both alike bear the semblance of divinity. —*Goethe*

(17) ________ is getting something right in language. —*Howard Nemerov*

See pg. 133 for answers.

These quotations are taken from an article by Professor Growney entitled "Mathematics and Poetry: Isolated or Integrated" which appeared in the *Humanistic Mathematics Network Newsletter* #6 (May 1991), 60–69. To subscribe contact Alvin White, Harvey Mudd College.