# HOMEWORK 1

Due in class *Mon, Jan. 29.*

**1.** (Bass 11.3) Let $m$ be Lebesgue measure on $\mathbb{R}$ and prove that for every measurable $f\colon \mathbb{R} \to \mathbb{R}$ we have

$$\int_{-\infty}^{\infty} |f(x)|\,dx = \int_0^{\infty} m(\{x : |f(x)| \geq t\})\,dt.$$

**2.** (Bass 11.6,10) Let $f \in L^1([0,1]^2, m)$, where $m$ is two-dimensional Lebesgue measure.

(a) Prove that if $\int_0^a \int_0^b f(x,y)\,dy\,dx = 0$ for all $a, b \in [0,1]$, then $f = 0$ $m$-a.e.

*Hint: one way to proceed is to let $\mathcal{C} = \{E \subset [0,1]^2 : \int_E f\,dm = 0\}$, prove that $\mathcal{C}$ contains all sets that are the direct product of two intervals, then prove that $\mathcal{C}$ contains all Borel sets and apply a general result.*

(b) Give an example where

$$\int_0^a \int_0^1 f(x,y)\,dy\,dx = 0 \quad \text{and} \quad \int_0^1 \int_0^b f(x,y)\,dy\,dx = 0$$

for all $a, b \in [0,1]$, but $f$ does not vanish $m$-a.e.

**3.** (Bass 11.11) Let $\mu$ be a finite measure on $\mathbb{R}$ and let $f(x) = \mu((-\infty, x])$. Given $c > 0$, show that $\int [f(x+c) - f(x)]\,dx = c\mu(\mathbb{R})$.

**4.** (Folland 2.59, Bass 11.12) Let $f(x) = \frac{\sin x}{x}$.

(a) Show that $\int_0^\infty |f(x)|\,dx = \infty$.

(b) Show that $\lim_{b\to\infty} \int_0^b f(x)\,dx = \frac{\pi}{2}$ by integrating $e^{-xy}\sin x$ over $(0,b) \times (0,\infty)$.