HOMEWORK 3

Due in class *Wed, Feb. 21.*

1. (Bass 13.5) Let μ, ν be two finite measures on a measurable space $(X, \mathcal{A})$. Say that μ and ν are *equivalent measures* if $\mu \ll \nu$ and $\nu \ll \mu$. Show that μ and ν are equivalent if and only if there exists $f \in L^1(\mu)$ such that f is strictly positive μ-a.e. and $d\nu = f\,d\mu$.

2. (adapted from Bass 13.11) Let $(X, \mathcal{F}, \mu)$ be a measure space, and $\mathcal{E} \subset \mathcal{F}$ a sub-σ-algebra of $\mathcal{F}$. Let f be a nonnegative function that is $\mathcal{F}$-measurable and μ-integrable. Define $\nu(A) = \int_A f\,d\mu$ for $A \in \mathcal{E}$, and let $\bar{\mu}$ be the restriction of μ to $\mathcal{E}$.

(a) Prove that $\nu \ll \bar{\mu}$ as measures on $(X, \mathcal{E})$, so the Radon–Nikodym derivative $g = d\nu/d\bar{\mu}$ exists and is $\mathcal{E}$-measurable. Show that $\int_A g\,d\mu = \int_A f\,d\mu$ for all $A \in \mathcal{E}$.

The function g is called the ***conditional expectation*** *of f with respect to $\mathcal{E}$, and written $g = \mathbb{E}[f|\mathcal{E}]$. If f is integrable and real-valued but not necessarily nonnegative, we define $\mathbb{E}[f|\mathcal{E}] = \mathbb{E}[f^+|\mathcal{E}] - \mathbb{E}[f^-|\mathcal{E}]$.*

(b) Show that if h is $\mathcal{E}$-measurable and $\int_A h\,d\mu = \int_A f\,d\mu$ for all $A \in \mathcal{E}$, then $h = g$ μ-a.e. Then show that $f = g$ μ-a.e. if and only if f is equal μ-a.e. to a $\mathcal{E}$-measurable function.

(c) Show that if $\mathcal{E} = \sigma\{A_1, A_2, \dots\}$, where $X = \bigsqcup_{n=1}^\infty A_n$ is a partition of X into disjoint sets A_n, then for every n with $\mu(A_n) > 0$, and μ-a.e. $x \in A_n$, we have

$$\mathbb{E}[f|\mathcal{E}](x) = \frac{1}{\mu(A_n)} \int_{A_n} f\,d\mu.$$

(d) Let $X = [0,1]^2$, let $\mathcal{F}$ be the Borel σ-algebra on X, and let $\mathcal{E} = \{\emptyset, [0,1]\} \times \mathcal{B}([0,1])$ be the σ-algebra containing all sets of the form $[0,1] \times B$, where $B \subset [0,1]$ is Borel. Let $f(x,y) = \sin(\pi x)\sin(\pi y)$, and determine $\mathbb{E}[f|\mathcal{E}]$.

3. (Bass 14.5) A real-valued function f is *Lipschitz* with constant M if $|f(x) - f(y)| \le M|x - y|$ for all $x, y \in \mathbb{R}$. Prove that f is Lipschitz with constant M if and only if f is absolutely continuous and $|f'| \le M$ a.e.

4. (Bass 14.6 and 14.9)

(a) Let f_n be a sequence of nondecreasing nonnegative functions on $[0,1]$; let $f = \sum_{n=1}^\infty f_n$ and suppose $f(1) < \infty$. Prove that $f'(x) = \sum_{n=1}^\infty f_n'(x)$ for a.e. x.

(b) Find a strictly increasing function $f\colon [0,1] \to \mathbb{R}$ such that $f' = 0$ a.e.

5. (Bass 14.10) If $f\colon [a,b] \to \mathbb{R}$ is continuous, let $M(y) = \#\{x \in [a,b] : f(x) = y\}$; note that this may be finite or infinite. Prove that M is a Borel measurable function and that $\int M(y)\,dy$ equals the total variation of f on $[a,b]$; in particular, if f is a continuous function of bounded variation then $M(y)$ is finite for a.e. y.