HOMEWORK 4

Due in class *Wed, Mar. 7.*

1. (Bass 15.21). Suppose $p > 1$ and q is its conjugate exponent, $f\colon [0,1] \to \mathbb{R}$ is absolutely continuous with $f' \in L^p$, and $f(0) = 0$. Prove that if $g \in L^q$, then

$$\int_0^1 |fg|\,dx \le p^{-1/p} \|f'\|_p \|g\|_q.$$

2. Given a normed vector space X and $x_n, x \in X$, say that $x_n \to x$ *weakly* if $H(x_n) \to H(x)$ for every bounded linear functional $H \in X^*$.

(a) Prove that norm convergence implies weak convergence: if $\|x_n - x\| \to 0$ then $x_n \to x$ weakly.

(b) (Bass 15.10, Folland 6.2.20). Fix $1 < p < \infty$, and suppose that $f_n \in L^p(\mu)$ is such that $\sup_n \|f_n\|_p < \infty$ and $f_n \to f$ μ-a.e.; prove that $f_n \to f$ weakly. *Hint: given $g \in L^q$, where q is conjugate to p, pass to a finite measure subset A such that $\int_{X\setminus A} |g|^q\,d\mu$ is small, then pass to a subset of A on which f_n converges uniformly.*

(c) Use the previous part to give an example of a sequence that converges weakly but not in norm.

3. (Bass 17.5). Let X be a compact metric space. Prove that $C(X)$ has a countable dense subset.

4. (Bass 17.7). Let X be a compact metric space and $\mathcal{B}$ its Borel σ-algebra. Let μ_n be a sequence of finite measures on $(X, \mathcal{B})$ with $\sup_n \mu_n(X) < \infty$.

(a) Prove that for every $f \in C(X)$ there is a subsequence n_j such that $\int f\,d\mu_{n_j}$ converges.

(b) Let $A \subset C(X)$ be a countable dense subset; prove that there is a subsequence n_j such that $\int f\,d\mu_{n_j}$ converges for all $f \in A$.

(c) With n_j as in the previous part, prove that $\int f\,d\mu_{n_j}$ converges for all $f \in C(X)$.

(d) Define $L\colon C(X) \to \mathbb{R}$ by $L(f) = \lim_{j\to\infty} \int f\,d\mu_{n_j}$. Prove that $L(f)$ is a positive linear functional on $C(X)$. Conclude that there is a measure μ on $(X, \mathcal{B})$ such that $\int f\,d\mu_{n_j} \to \int f\,d\mu$ for all $f \in C(X)$. *We say that μ is the **weak* limit** of the sequence μ_{n_j}; this exercise proves that for every $K > 0$, the space of finite measures on $(X, \mathcal{B})$ with $\mu(X) \le K$ is compact in the weak* topology.*