HOMEWORK 5

Due in class *Wed, Mar. 28.*

1. (Bass 17.6) Let X be a compact metric space and $\mathcal{B}$ its Borel σ-algebra. Let μ_n, μ be finite Borel measures on X such that $\mu_n(X) \to \mu(X)$. Prove that the following are equivalent:

(a) $\int f\,d\mu_n \to \int f\,d\mu$ for all $f \in C(X)$;

(b) $\limsup_{n\to\infty} \mu_n(F) \le \mu(F)$ for all closed $F \subset X$;

(c) $\liminf_{n\to\infty} \mu_n(G) \ge \mu(G)$ for all open $G \subset X$;

(d) $\lim_{n\to\infty} \mu_n(A) = \mu(A)$ whenever $A \subset X$ is Borel and has $\mu(\partial A) = 0$, where $\partial A = \overline{A} \cap \overline{A^c}$ is the boundary of A.

2. (adapted from Bass 18.4) Let $\alpha \in (0,1)$. Given $f \in C([0,1])$, define

$$|f|_\alpha = \sup_{x,y\in[0,1],x\neq y} \frac{|f(x)-f(y)|}{|x-y|^\alpha}, \qquad \|f\|_\alpha = \|f\| + |f|_\alpha.$$

Let $C^\alpha([0,1]) = \{f \in C(X) : \|f\|_\alpha < \infty\}$. Prove that $\|f\|_\alpha$ is a norm on $C^\alpha([0,1])$. Is $C^\alpha([0,1])$ complete with respect to this norm?

3. (Folland 5.25) Suppose that X is a Banach space and that X^* is separable. Prove that X is separable. *Hint: Take a dense sequence $f_n \in X^*$ and produce $x_n \in X$ such that $\|x_n\| = 1$ and $|f_n(x_n)| \ge \frac{1}{n}\|f_n\|$; then show that linear combinations of the x_n's are dense in X.*

4. (Bass 18.11) Suppose that $\|\cdot\|_1$ and $\|\cdot\|_2$ are two norms on a vector space X such that $\|x\|_1 \le \|x\|_2$ for all $x \in X$, and that X is complete with respect to both norms. Prove that there is $c > 0$ such that $\|x\|_2 \le c\|x\|_1$ for all $x \in X$.

5. (Bass 18.12) Suppose X and Y are Banach spaces.

(a) Prove that $X \times Y$ is a Banach space with the norm $\|(x,y)\| = \|x\| + \|y\|$.

(b) Let $L\colon X \to Y$ be a linear map such that if $x_n \to x$ in X and $Lx_n \to y$ in Y, then $y = Lx$. Such a map is called a *closed map.* Let $G = \{(x, Lx) : x \in X\} \subset X \times Y$ be the *graph* of L. Prove that G is a closed subset of $X \times Y$, and hence is complete.

(c) Prove that the function $(x, Lx) \mapsto x$ is continuous, injective, linear, and maps G onto X.

(d) Prove the *closed graph theorem*: if X, Y are Banach spaces and $L\colon X \to Y$ is a closed linear map, then L is continuous.