HOMEWORK 6

Due in class *Wed, Apr. 11.*

1. Let H be a Hilbert space and $M \subset H$ a subspace. Prove that $(M^\perp)^\perp = \overline{M}$.

2. (Bass 19.9). Prove that if $A \subset [0, 2\pi]$ is measurable, then $\lim_{n\to\infty} \int_A e^{inx}\,dx = 0$. (This is a special case of the *Riemann-Lebesgue Lemma*).

3. (Folland 5.57). Let H be a Hilbert space and $T \in L(H, H)$ a bounded linear operator from H to itself.

(a) Prove that there is a unique $T^* \in L(H, H)$ such that $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for all $x, y \in H$. (T^* is called the *adjoint* of T.)

(b) Prove that $\|T^*\| = \|T\|$, $\|T^*T\| = \|T\|^2$, $(aS + bT)^* = \overline{a}S^* + \overline{b}T^*$, $(ST)^* = T^*S^*$, and $T^{**} = T$.

(c) Prove that $T(H)^\perp = \ker(T^*)$ and $\ker(T)^\perp = \overline{T^*(H)}$.

(d) Prove that the following are equivalent: (i) T is invertible and $\langle Tx, Ty\rangle = \langle x, y\rangle$ for all $x, y \in H$; (ii) T is invertible and $T^{-1} = T^*$. In this case T is called *unitary*.

4. (Bass 19.13). Suppose $\{e_n\}$ is an orthonormal basis for a separable Hilbert space and $\{f_n\}$ is an orthonormal set such that $\sum \|e_n - f_n\| < 1$. Prove that $\{f_n\}$ is a basis.

5. (Folland 5.63). Let H be an infinite-dimensional Hilbert space. Recall that we say that a sequence x_n in H converges weakly to $x \in H$ if $\langle x_n, y\rangle \to \langle x, y\rangle$ for all $y \in H$.

(a) Prove that every orthonormal sequence in H converges weakly to 0.

(b) Prove that the unit sphere $S = \{x : \|x\| = 1\}$ is weakly dense in the unit ball $B = \{x : \|x\| \le 1\}$.

6. (Bass 19.11). Suppose f is a continuous real-valued function on $\mathbb{R}$ such that $f(x+1) = f(x)$ for every x. Let γ be an irrational number. Prove that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} f(j\gamma) = \int_0^1 f(x)\,dx.$$