HOMEWORK 1

Due in class *Wed, Jan. 31.*

1. (BG 1.15.1) Let S^2 be the unit sphere in $\mathbb{R}^3$, and let $\mathbf{n}, \mathbf{s} \in S^2$ be the north and south poles $(0, 0, \pm 1)$. Let $\phi_1 \colon \mathbb{R}^2 \to S^2$ be the *stereographic projection towards the north pole* defined as follows: given $x \in \mathbb{R}^2$, let $\ell \subset \mathbb{R}^3$ be the line through $(x_1, x_2, -1)$ and $\mathbf{n}$, and let $\phi_1(x)$ be the point at which ℓ intersects S^2 (apart from $\mathbf{n}$). Similarly let $\phi_2 \colon \mathbb{R}^2 \to S^2$ be *sterographic projection towards the south pole*, defined analogously but with ℓ being the line through $(x_1, x_2, 1)$ and $\mathbf{s}$.

(a) Write explicit formulas for ϕ_1 and ϕ_2.

(b) Compute the transition map $\phi_1^{-1} \circ \phi_2$ (specify its domain as well) and demonstrate that ϕ_1, ϕ_2 give a smooth atlas for S^2.

Remark: stereographic projection similarly gives a two-chart smooth atlas for the n-sphere $S^n \subset \mathbb{R}^{n+1}$.

2. (Lee 1.9, BG 1.15.4) Complex projective n-space $\mathbb{C}P^n$ is the set of all *complex* lines in $\mathbb{C}^{n+1}$; that is, it is the quotient space of $\mathbb{C}^{n+1} \setminus \{0\}$ by the equivalence relation "$z \sim cz$ for all $c \in \mathbb{C} \setminus \{0\}$".

(a) Show that $\mathbb{C}P^n$ can be made into a compact $2n$-dimensional smooth manifold using a smooth atlas that is analogous to the one we constructed for $\mathbb{R}P^n$. (Once you have described the charts, you must compute the transition maps.)

(b) Show that $\mathbb{C}P^1$ and S^2 are diffeomorphic.

Note that we identify $\mathbb{C}^k$ with $\mathbb{R}^{2k}$ via $(x_1 + iy_1, \ldots x_k + iy_k) \leftrightarrow (x_1, y_1, \ldots, x_k, y_k)$.

3. Recall that one description of the torus $\mathbb{T}^2$ was as "the unit square with opposite edges identified by translation". In other words, writing X for the unit square and $\sim$ for the equivalence relation on X that identifies $(x, 0) \sim (x, 1)$ and $(0, y) \sim (1, y)$, the quotient space $X/{\sim}$ carries a natural smooth structure making it diffeomorphic to the surface of revolution in $\mathbb{R}^3$ that was our original model for the torus.

(a) Let X be a regular hexagon and $\sim$ the equivalence relation that identifies opposite sides of X by translation. Describe a smooth structure on $X/{\sim}$, and sketch and describe a regular surface in $\mathbb{R}^3$ that is diffeomorphic to $X/{\sim}$. (You do not need to write a formula for the surface or for the diffeomorphism, a rough description is enough.)

(b) Do the same thing when X is a regular octagon.

4. (BG 1.15.12) Give an example of a smooth homeomorphism that is not a diffeomorphism, illustrating that the condition that the inverse mapping be smooth is independent.