# HOMEWORK 2

Due in class *Mon, Feb. 12.*

**1.** (Lee, 2-9) Let $p$ be a non-zero polynomial in one variable with complex coefficients. Show that there is a unique continuous map $\tilde{p}\colon \mathbb{C}P^1 \to \mathbb{C}P^1$ such that the following diagram commutes, where $G\colon \mathbb{C} \to \mathbb{C}P^1$ is given by $G(z) = [z, 1]$:

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{G} & \mathbb{C}P^1 \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \tilde{p}} \\ \mathbb{C} & \xrightarrow{G} & \mathbb{C}P^1 \end{array}$$

**2.** (BG 1.15.13)

(a) Let $M$ and $N$ be smooth manifolds, and prove that $T(M \times N)$ is diffeomorphic to $TM \times TN$; that is, the tangent bundle of a product is diffeomorphic to the product of the tangent bundles.

(b) Prove that $TS^1$ is diffeomorphic to $S^1 \times \mathbb{R}$, then use part (a) to show that $T\mathbb{T}^2$ is diffeomorphic to $\mathbb{T}^2 \times \mathbb{R}^2$.

**3.** Let $M$ be an oriented embedded surface in $\mathbb{R}^3$. In example 1.7.3 of the textbook, the *Gauss map* $M \to S^2$ is defined as follows: given a point $p \in M$ and a local parametrization $\phi\colon U \to V \ni p$ from our oriented smooth atlas, we put

$$F(p) = \frac{\partial \phi}{\partial x_1}(\phi^{-1}p) \times \frac{\partial \phi}{\partial x_2}(\phi^{-1}p) \in \mathbb{R}^3,$$

where $\times$ represents the cross product in $\mathbb{R}^3$, and then we define $G\colon M \to S^2$ by

$$G(p) = F(p)/\|F(p)\|.$$

Note that $T_pM$ and $T_{G(p)}S^2$ are naturally identified with the same two-dimensional subspace of $\mathbb{R}^3$, and so $dG_p\colon T_pM \to T_{G(p)}S^2$ can be viewed as a linear map from $T_pM$ to itself. The determinant of this map is called the *Gaussian curvature* of $M$ at $p$.

(a) Prove that $F(p) \neq 0$ for all $p \in M$, so that $G(p)$ makes sense. Then prove that $G(p)$ does not depend on which parametrization we choose near $p$.

(b) Prove that if $M$ is compact, then $G\colon M \to S^2$ is surjective, and that there must be some point $p$ where the Gaussian curvature is nonnegative.

(c) Consider the surface $M$ given by $z = x^2 - y^2$, identify $T_0M$ with $\mathbb{R}^2$ using the natural identification from the chart $\phi(x, y) = (x, y, x^2 - y^2)$, and compute the matrix for the linear map $dG_0\colon \mathbb{R}^2 \to \mathbb{R}^2$. Use this to find the Gaussian curvature of $M$ at 0.