HOMEWORK 3

Due in class *Mon, Feb. 26.*

1. (BG 1.15.14) Show that any injective immersion of a compact manifold is an embedding.

2. (BG 1.15.15) Show that if an embedded surface $S \subset \mathbb{R}^3$ is given by $S = f^{-1}(q)$, where $f\colon \mathbb{R}^3 \to \mathbb{R}$ is a smooth function and q is a regular value for f, then S is orientable.

3. (BG 1.15.17) Consider the projective place $\mathbb{R}P^2 = S^2/\sim$, where $S^2 \subset \mathbb{R}^3$ is the unit sphere and $p \sim -p$. Define $f\colon S^2 \to \mathbb{R}^4$ by

$$f(x, y, z) = (x^2 - y^2, xy, xz, yz).$$

Note that $f(p) = f(-p)$ for all $p \in S^2$, so we can define $F\colon \mathbb{R}P^2 \to \mathbb{R}^4$ by $F([p]) = f(p)$. Prove that F is an embedding.

4. (a) Prove that given any closed set $C \subset \mathbb{R}$, there is a smooth function $f\colon \mathbb{R} \to \mathbb{R}$ such that $f^{-1}(0) = C$.

(b) (BG 1.15.19, modified) Prove that given any closed set $C \subset \mathbb{R}$, there exists a submanifold $M \subset \mathbb{R}^2$ such that $M \cap (\mathbb{R} \times \{0\}) = C \times \{0\}$. Show that if M is any such manifold with this property, then M is not transverse to $\mathbb{R} \times \{0\}$. *(This shows that if two intersecting manifolds are not transverse, then their intersection may fail to be a submanifold.)*