HOMEWORK 5

Due in class *Mon, Apr. 2.*

1. (BG 3.4.1). If (M_1, g_1) and (M_2, g_2) are Riemannian manifolds, show that the mapping g defined by $g_{(p_1,p_2)}((X_1, X_2), (Y_1, Y_2)) = (g_1)_{p_1}(X_1, Y_1) + (g_2)_{p_2}(X_2, Y_2)$ defines a Riemannian metric on $M_1 \times M_2$, called the product metric.

2. (BG 3.4.2). Consider the Riemannian metric induced by $\mathbb{R}^4$ on the torus $S^1 \times S^1$, parametrized as $(\cos(x_1), \sin(x_1), \cos(x_2), \sin(x_2))$. Show that $S^1 \times S^1$ with the induced metric is isometric to the flat torus $\mathbb{R}^2/\mathbb{Z}^2$.

3. (BG 3.4.8). Consider the Poincaré half plane $\mathbb{H}^2$ and the Poincaré disc $\mathbb{B}^2$.
(a) Show that the mapping $f\colon \mathbb{H}^2 \to \mathbb{B}^2$ defined by $f(z) = (i - z)/(i + z)$ is an isometry.
(b) Consider the manifold $\mathcal{H}^2$ consisting of the upper sheet ($z > 0$) of the hyperboloid $x^2 + y^2 - z^2 = -1$ with the smooth structure induced by $\mathbb{R}^3$. Endow this manifold with the Minkowski metric $ds^2 = dx^2 + dy^2 - dz^2$. This is called the hyperboloid model of the hyperbolic plane. Show that the map $g\colon \mathbb{H}^2 \to \mathcal{H}^2$ given by

$$g(z) = \frac{(2\,\mathrm{Re}(z), 2\,\mathrm{Im}(z), 1 + |z|^2)}{(1 - |z|^2)}$$

is an isometry.

4. (BG 4.8.1). Consider $\mathbb{R}$ with the connection $\nabla_{(\partial/\partial x)}(\partial/\partial x) = \lambda$, for some $\lambda \in \mathbb{R}$. Let $c\colon [0, 1] \to \mathbb{R}$ be a curve with $dc/dt(0) = \partial/\partial x$. Show that the parallel transport along c

$$P_{c(t),c(0)}\colon T_{c(0)}\mathbb{R} \to T_{c(t)}\mathbb{R}$$

is given by

$$P_{c(t),c(0)}(v\partial/\partial x) = ve^{-\lambda t}(\partial/\partial x),$$

for $v \in \mathbb{R}$. Note that $\lambda = 0$ gives the usual connection, and every $\lambda \neq 0$ determines a non-Euclidean parallelism on $\mathbb{R}$.