HOMEWORK 6

Due in class *Fri, Apr. 20.*

This assignment walks you through the proof of part of the *Gauss–Bonnet theorem.* The goal is to prove a result that relates curvature of a surface to angles in geodesic triangles (#4), and then use this to prove one specific case of Gauss–Bonnet in #5.

Let M be a two-dimensional Riemannian manifold. Given $p \in M$, fix $v_0 \in T_pM$ with $\|v_0\| = 1$, and for each $\theta \in \mathbb{R}$, let v_θ be the unit vector in T_pM that makes an angle θ with v_0. (This involves a choice of orientation; it does not matter which way we rotate, just pick one and stick with it.) Define *geodesic polar coordinates* around p by the coordinate map $\phi\colon (0,\infty) \times \mathbb{R} \to M$ by $\phi(r,\theta) = \exp_p(rv_\theta)$. It follows from properties of the exponential map that ϕ is a local diffeomorphism when r is sufficiently small.

1. Prove that in geodesic polar coordinates, the Riemannian metric takes the form $ds^2 = dr^2 + g(r,\theta)^2 d\theta^2$ for some positive smooth function g, and that $\lim_{r\to 0^+} \frac{\partial g}{\partial r} = 1$. Compute the Christoffel symbols for this coordinate system (here $x_1 = r$ and $x_2 = \theta$), and use these to compute the vector fields $\nabla_{\frac{\partial}{\partial r}} \frac{\partial}{\partial r}$, $\nabla_{\frac{\partial}{\partial r}} \frac{\partial}{\partial \theta}$, $\nabla_{\frac{\partial}{\partial \theta}} \frac{\partial}{\partial r}$, and $\nabla_{\frac{\partial}{\partial \theta}} \frac{\partial}{\partial \theta}$. You will need the fact that if g_{ij} represents the metric in local coordinates, and $g^{k\ell}$ is the inverse of the matrix g_{ij}, then

$$\Gamma_{ij}^k = \frac{1}{2} \sum_{\ell=1}^m \Big(\frac{\partial g_{i\ell}}{\partial x_j} + \frac{\partial g_{j\ell}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_\ell} \Big) g^{k\ell} \quad \text{and} \quad \nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \sum_{k=1}^m \Gamma_{ij}^k \frac{\partial}{\partial x_k}$$

2. Recall that the *Gaussian curvature* of M at a point p is the real number given by

$$\kappa(p) = \frac{\langle (R(X,Y)Y)_p, X_p \rangle}{\|X_p\|^2 \|Y_p\|^2 - \langle X_p, Y_p \rangle},$$

where X, Y are (any) vector fields with X_p, Y_p linearly independent, and R is the curvature tensor defined by $R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$. Using #1, prove that in geodesic polar coordinates, the Gaussian curvature is given by

$$\kappa(r,\theta) = -\frac{1}{g} \frac{\partial^2 g}{\partial r^2}.$$

3. Let γ be a curve given in geodesic polar coordinates by $\gamma(t) = \phi(r(t), \theta(t))$. Prove that γ is a geodesic if and only if the following equations are satisfied:

$$\frac{d^2 r}{dt^2} - g \frac{\partial g}{\partial r} \Big(\frac{d\theta}{dt} \Big)^2 = 0,$$

$$\frac{d^2\theta}{dt^2} + \frac{2}{g} \frac{\partial g}{\partial r} \frac{dr}{dt} \frac{d\theta}{dt} + \frac{1}{g} \frac{\partial g}{\partial \theta} \Big(\frac{d\theta}{dt} \Big)^2 = 0.$$

4. We define integration on a Riemannian manifold M with respect to volume as follows: given a region $E \subset M$ that is covered by a single chart ϕ, we write

$$\int_E f(p)\,dV(p) = \int_{\phi^{-1}(E)} f(\phi(x_1, \ldots, x_m))\sqrt{|\det g_{ij}(x_1, \ldots, x_m)|}\,dx_1 \cdots dx_m.$$

When M is two-dimensional, as in the present exercise, we refer to volume as surface area and write dS (or dA) in place of dV. Let E be a geodesic triangle – that is, E is the region bounded by three geodesics on M, that intersect in points $A, B, C \in M$, where they make angles α, β, γ. Suppose that E is covered by a single set of geodesic polar coordinates, and prove that

$$\int_E \kappa\,dS = \alpha + \beta + \gamma - \pi.$$

Hint: Use geodesic polar coordinates centered at A and let $\omega\colon [b, c] \to M$ be the unit speed geodesic from B to C. Compute the derivative of $\zeta(t) = \angle(\frac{d\omega}{dt}, \frac{\partial}{\partial r})$ using #3, and relate $\zeta(b)$, $\zeta(c)$ to β and γ. Then write $\int_E \kappa\,dS$ as a double integral in these coordinates, and simplify as much as possible using #2. Relate the two computations to complete the proof.

Remark: This says that the *angular excess* of the triangle E is the total curvature that it encloses, so triangles enclosing positive total curvature will have angles adding up to more than π (as with great circles on a sphere), while triangles enclosing negative total curvature will have angles adding up to less than π (as with triangles in the hyperbolic plane). Note that if we move a vector by parallel transport around the perimeter of the triangle, the result is a rotation of the original vector, determined by the angular excess, and hence by the total curvature.

5. Prove that if g is any Riemannian metric on the torus $\mathbb{T}^2$, then $\int_{\mathbb{T}^2} \kappa\,dS = 0$.

Hint: Consider the torus as $[0, 1]^2$ with opposite edges identified, and partition it into a large number of small geodesic triangles with vertices at the points $(\frac{i}{n}, \frac{j}{n})$ for $0 \leq i, j \leq n$, where n is chosen large enough that the necessary geodesics connecting nearby points all exist. Then show that the angular excesses of these triangles must add up to 0 by counting how many total vertices and edges your partition has.

Remark: One can similarly prove that for any Riemannian metric on the sphere S^2, we have $\int_{S^2} \kappa\,dS = 4\pi$, and that if M is the **surface of genus** k defined by identifying opposite edges of a regular $4k$-gon, then any Riemannian metric on M has $\int_M \kappa\,dS = 4\pi(1 - k)$. This is usually written as $\int_M \kappa\,dS = 2\pi\chi(M)$, where $\chi(M) = 2 - 2k$ is the **Euler characteristic**. The sphere has Euler characteristic 2, and the process of "adding a handle" reduces Euler characteristic by 2. Euler characteristic is typically defined by 'triangulating' the surface as in #5, then putting

$$\chi(M) = \#\text{vertices} - \#\text{edges} + \#\text{faces}.$$