HOMEWORK 7

Due in class *Mon, Apr. 30.*

1. (Lee 11.7a). Let $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the smooth map given by $F(s,t) = (st, e^t)$, and let $\omega \in \mathfrak{X}^*(\mathbb{R}^2)$ be the covector field given by $\omega = x\,dy - y\,dx$. Compute $F^*\omega$.

2. (Lee 11.14). Consider the following two covector fields on $\mathbb{R}^3$:

$$\omega = -\frac{4z\,dx}{(x^2+1)^2} + \frac{2y\,dy}{y^2+1} + \frac{2x\,dz}{x^2+1},$$
$$\eta = -\frac{4xz\,dx}{(x^2+1)^2} + \frac{2y\,dy}{y^2+1} + \frac{2\,dz}{x^2+1}.$$

(a) Set up and evaluate the line integral of each covector field along the straight line segment from $(0,0,0)$ to $(1,1,1)$.
(b) Determine whether either of these covector fields is exact.
(c) For each one that is exact, find a potential function and use it to recompute the line integral.

3. (BG 6.11.1). Prove that TM and T^*M are isomorphic as vector bundles; that is, there is a diffeomorphism $f\colon TM \to T^*M$ such that f restricts to a linear isomorphism from T_pM to T_p^*M for all $p \in M$. *Hint: use a Riemannian metric.*

4. Given a basis $\{e_1, \ldots, e_n\}$ for a vector space V, the *dual basis* for V^* is the set $\{L_1, \ldots, L_n\}$ defined by $L_i(e_j) = \delta_{ij}$. A $\binom{k}{0}$*-tensor* on a V is a multilinear map $V^k \to \mathbb{R}$. The *tensor product* of a $\binom{k}{0}$-tensor ω and a $\binom{\ell}{0}$-tensor η is the $\binom{k+\ell}{0}$-tensor

$$\omega \otimes \eta(v_1, \ldots, v_k, v_{k+1}, \ldots, v_{k+\ell}) = \omega(v_1, \ldots, v_k)\eta(v_{k+1}, \ldots, v_{k+\ell}).$$

Consider the $\binom{2}{0}$-tensor det on $\mathbb{R}^2$ given by $\det = L_1 \otimes L_2 - L_2 \otimes L_1$. Determine, with proof, whether or not there are covectors ($\binom{1}{0}$-tensors) ω, η on $\mathbb{R}^2$ such that $\det = \omega \otimes \eta$.

5. Let $\omega_1, \ldots, \omega_k$ be covectors on a finite-dimensional vector space V.
(a) (Lee 14.1). Show that $\omega_1, \ldots, \omega_k$ are linearly dependent iff $\omega_1 \wedge \cdots \wedge \omega_k = 0$.
(b) Suppose $\omega_1, \ldots, \omega_k$ are linearly independent, and so is the collection of covectors $\eta_1, \ldots, \eta_k \in V^*$. Prove that $\operatorname{span}(\omega_1, \ldots, \omega_k) = \operatorname{span}(\eta_1, \ldots, \eta_k)$ if and only if there is some nonzero real number c such that $\omega_1 \wedge \cdots \wedge \omega_k = c\,\eta_1 \wedge \cdots \wedge \eta_k$.

6. (Lee 16-2). Let $\mathbb{T}^2 = S^1 \times S^1 \subset \mathbb{R}^4$ denote the 2-torus, defined as the set of points (w,x,y,z) such that $w^2 + x^2 = y^2 + z^2 = 1$, with the product orientation determined by the standard orientation on S^1. Consider the 2-form $\omega = xyz\,dw \wedge dy \in \Omega^2(\mathbb{R}^4)$ and compute $\int_{\mathbb{T}^2} \omega$.