HW #1. Due *Wednesday, February 27.*

(1) Prove that the full two-shift has a point whose orbit is dense. (Describe such a point as explicitly as you can.)

(2) Prove the following shadowing property for the doubling map $f\colon S^1 \to S^1$: for every $\epsilon > 0$, there exists $\delta > 0$ such that if the sequence $x_0, x_1, x_2, \dots \in S^1$ satisfies $d(f(x_n), x_{n+1}) < \delta$ for all n (a δ-pseudo-orbit), then there exists a unique $y \in S^1$ such that $d(f^n y, x_n) < \epsilon$ for all n.

(3) Prove that the doubling map has the following *specification property*: for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all pairs $(x_1, n_1), \dots, (x_k, n_k) \in S^1 \times \mathbb{N}$ and for all integers $N_1, \dots, N_k$ satisfying $N_{i+1} - (N_i + n_i) \geq N$ $(i = 1, \dots, k-1)$, there exists $y \in S^1$ such that for all i, we have $d(f^j(f^{N_i} y), f^j x_i) < \epsilon$ for all $0 \leq j \leq n_i$.
Hint: First construct a pseudo-orbit, then apply the result of the previous problem.

(4) Describe all of the ergodic measures for the map $f(x, y) = (x + y, y) \bmod \mathbb{Z}^2$ on the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. Is the set of ergodic measures dense in $\mathcal{M}_f$?

(5) Prove that for the full two-shift, the set of periodic orbit measures is dense in the set of all invariant measures $\mathcal{M}_\sigma$.
Hint: First use the ergodic decomposition to prove that every invariant measure μ can be approximated by a (finite) convex combination of ergodic measures. Then find generic points for each of these measures, and use properties of Σ_2^+ to find a single periodic point y whose trajectory shadows each of these generic points for the right length of time, so that the corresponding measure is close to μ.