Math 7394 Ergodic theory and thermodynamic formalism Spring 2019

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HW #2. Due *Wednesday, April 3.*

(1) Let $f\colon S^1 \to S^1$ be a *uniformly expanding* continuous map, meaning that there are $\lambda > 1$ and $\epsilon_0 > 0$ such that if $d(x, y) < \epsilon_0$, then $d(fx, fy) \geq \lambda d(x, y)$. Prove the following shadowing property for f: for every $\epsilon > 0$, there exists $\delta > 0$ such that if the sequence $x_0, x_1, x_2, \dots \in S^1$ satisfies $d(f(x_n), x_{n+1}) < \delta$ for all n (a δ-pseudo-orbit), then there exists a unique $y \in S^1$ such that $d(f^n y, x_n) < \epsilon$ for all n.

This is of course extremely similar to the corresponding problem for the doubling map from HW1, but if you gave an algebraic solution there, you may find some trouble in generalizing it. A better approach is to generalize the solution that I gave you.

(2) Let $(X, \mathcal{B}, \mu)$ be a probability space, and let $f\colon X \to X$ preserve μ. Prove that

$$h_\mu(f) = \sup\{h_\mu(f, \alpha) : \alpha \text{ is a finite measurable partition of } X\};$$

in other words, the definition of entropy gives the same value if we only consider finite partitions (instead of also allowing countable partitions with finite entropy).

(3) Let $(X, \mathcal{B}, \mu, f)$ be an ergodic probability measure-preserving system. Let $Y \subset X$ be measurable with $\mu(Y) > 0$, and let $g\colon Y \to Y$ be the first return map to Y; that is, $g(x) = f^{\tau(x)}(x)$, where $\tau(x) = \min\{t \in \mathbb{N} : f^t(x) \in Y\}$. Let ν be the normalized restriction of μ to Y. Prove that ν is g-invariant and that $h_\mu(f) = \mu(Y)h_\nu(g)$.

(4) Let S be a finite alphabet and $X \subset S^{\mathbb{N}}$ a closed set with $\sigma(X) = X$, where σ is the shift map. Let $\mathcal{L}_n = \{w \in S^n : [w] \cap X \neq \emptyset\}$ be the set of words of length n that appear in some element of X, and $\mathcal{L} = \bigcup_{n=0}^\infty \mathcal{L}_n$.

(a) Prove that (X, σ) has the *specification property* stated in Problem 3 of the last homework if and only if the following is true: there exists $\tau \in \mathbb{N}$ such that for every $v, w \in \mathcal{L}$ there exists $u \in \mathcal{L}_\tau$ such that $vuw \in \mathcal{L}$.

(b) Let $d = \#S$, let T be a primitive $d \times d$ matrix of 0s and 1s, and let (X, σ) be the associated SFT. Prove that (X, σ) has the specification property.

(c) Suppose that (X, σ) has the specification property, and let $h = h_{\text{top}}(X, \sigma) = \lim_{n\to\infty} \frac{1}{n} \log \#\mathcal{L}_n$. Prove that there exists a constant $C \geq 1$ such that for every $n \in \mathbb{N}$, we have $e^{nh} \leq \#\mathcal{L}_n \leq Ce^{nh}$.