Real numbers, continued fractions, and rational approximations

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1 What is a real number?

One of the most enduring memories of my undergraduate education is also the very first. I was enrolled in Math 147, the advanced section of the standard first-year calculus course at the University of Waterloo, taught by Ken Davidson. On that first day, the classroom was bursting at the seams, with a hundred or so cocky young frosh crowded into a space meant for eighty. We had all been the brightest kids in our respective high schools, if not in everything, then certainly insofar as math was concerned, and we had not yet been hit with the healthy dose of humility that university-level mathematics courses would provide.

Over the course of the next four months, Professor Davidson would do a rather thorough job of cutting us down to size, and teaching us a thing or two about how mathematics is done in the process. It began with his opening salvo on that first day; as we sat there, confident in our ability to handle anything he could throw at us (after all, most of us had already seen some calculus in high school), he stepped to the front of the room, turned to face us, and asked a deceptively simple question: "What is a real number?"

After half an hour of fishing around and putting forward various suggestions, none of which really held water, we had come up with quite a few examples of how *not* to define a mathematical object, but still hadn't answered the question to his satisfaction. Professor Davidson was leading us towards the idea that before we can use anything in mathematics, we must first give it a proper definition, that the real numbers in fact need to be *constructed* from the building blocks of the integers.

As anyone who has gone through a course covering the foundations of the real line knows, this is a significant undertaking, particularly for students fresh out of high school, who have yet to see *real* mathematics. In the end, the best we could come up with was the idea of a decimal expansion; a real number is simply something which can be written as a (possibly infinite) decimal number.

That concept was sufficient for a working definition, and to get us out of that first class, but there are enough problems with the idea of representing real numbers by decimals that over the course of the term, we were introduced to the two usual methods of constructing the real numbers— namely, Cauchy sequences and Dedekind cuts. Both of these begin by constructing the rationals from the integers (which, as Kronecker would have it, are God-given); it is not the purpose of these brief notes to discuss either construction in detail,[1] but rather to examine a fourth way of thinking about real numbers. The concept is that of continued fractions, which describe each real number in terms of a series of positive integers. First, of course, we must answer the question of why we would bother to do this, when we already have three other perfectly valid ways of defining $\mathbb{R}$.

[1] If you have not seen these constructions before, or if you have no idea what I'm talking about in these first two sections, feel free to skip directly to section 3. Everything up to that point is simply motivation. If you really want to read more about them, have a look at Rudin [6] or Conway [2].

2 Why continued fractions?

Each of the usual three methods of constructing the real numbers—decimal expansions, Cauchy sequences, and Dedekind cuts—has its own particular idiosyncrasies, its own strengths and weaknesses.

To most of us, decimal expansions are the most familiar, and indeed, are the only one of the three which any non-mathematician is likely to have any contact with. We proclaim, as if by edict, “A real number is a sequence, possibly infinite, of digits from 0 to 9, with a decimal point placed somewhere,” and then define rules for working with them. If we are feeling particularly inspired, we might draw part of a number line, divide it into intervals of unit length, divide each of these into ten equal subintervals, and indicate that this is to be continued *ad infinitum*, thus suggesting a connection between the intuitive concept of points on a line and the real numbers which are to be represented. However, we cannot avoid the fact that if we had twelve fingers instead of ten, the construction would probably look somewhat different, and that the algorithms for adding and multiplying decimals work right-to-left, a difficulty when the expansion continues to the right indefinitely. Further, there is no particularly deep mathematics in a number’s decimal expansion; the expansion will tell us if the number is rational, but nothing more, and the arithmetic properties of the number are almost completely divorced from its decimal expansion.

Cauchy sequences, on the other hand, are very natural; further, the process by which the rationals are completed to obtain the reals can be easily generalised to any metric space,[2] and hence is a very powerful procedure. For this reason, it could be argued that Cauchy sequences provide the best construction of the real line; however, they have the disadvantage of being highly non-unique. Given any real number α, not only are there multiple sequences of rationals which converge to α, there are *uncountably many* such sequences. This makes it impossible to talk meaningfully about *the* Cauchy sequence which represents a particular real number, and we would like something a little more concrete and specific.

Dedekind cuts[3] do not suffer from the arbitrariness or non-uniqueness of the previous two approaches, but there is generally no good way to represent them. While the set of rational numbers less than $\sqrt{2}$ can be written as $\{x \in \mathbb{Q} : x \leq 0 \text{ or } x^2 < 2\}$, it would be difficult to conceive of a similar criterion for the relation $x < \pi$.

In contrast with these three constructions, continued fractions are quite natural, provide (very nearly) unique representations of real numbers, and can be quite easily written down, as they are simply sequences of integers. However, the one difficulty inherent in dealing with continued fractions is that they are nearly impossible to do calculations with. The task of adding or multiplying two continued fractions is so intractable as to make dividing Roman numerals look positively enjoyable.

[2] See exercises 24-25 in chapter 3 of [6] for details.

[3] See the appendix to chapter 1 of [6] for details of this construction; also, Conway [2] discusses a generalisation of Dedekind cuts, which also incorporates von Neumann’s construction of the ordinals into a unified system which he refers to as the *surreal numbers*.

Nevertheless, since we will be concerned here only with the continued fraction representation of single real numbers, this difficulty will not trouble us at the present time. As we will see, the great utility of continued fractions in finding rational approximations to real numbers makes them important mathematical objects in their own right. Further, as a fringe benefit, they allow certain constants, such as e and π, to be written in particularly nice forms.

In the interests of completeness (no pun intended), it ought to pointed out that both decimal expansions and continued fractions are a specialisation of the method of Cauchy sequences, in which one particular Cauchy sequence is singled out as "the" representative sequence for a real number α. For decimal expansions, it is the sequence which has the form

$$\left(\frac{a_0}{1}, \frac{a_1}{10}, \frac{a_2}{100}, \ldots, \frac{a_n}{10^n}, \ldots\right)$$

where $a_n \in \mathbb{Z}$, and each term is the largest rational number with that form which is still less than or equal to α. In the case of continued fraction expansions, we will see in due course which particular sequence we choose; the point here is that the underlying construction is still the idea of Cauchy sequences, so that in the end, all of the constructions discussed fall into the rubric of either Cauchy sequences or Dedekind cuts.

It is also worth noting that the image given above for decimal expansions, in which each unit interval is subdivided into ten equal subintervals, also applies to continued fractions. However, in this case each interval will be divided into *infinitely many* subintervals, which must necessarily have different lengths. It will be a worthwhile exercise to draw or visualise the geometry of the situation as the exposition proceeds.

3 What are continued fractions?

We know what fractions are: "two-level" numbers of a sort, with a numerator and a denominator. Now if the denominator itself contains another fraction, we have something like the following:

$$1 + \frac{1}{2 + \frac{1}{2}}$$

This sort of behaviour could continue (hence the name); we could insert another fraction into the innermost denominator:

$$1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2}}}$$

These numbers can be simplified, of course, and written in the usual manner; the former is equal to $\frac{7}{5}$, the latter to $\frac{17}{12}$. For the time being, though, let us agree to leave them in this somewhat awkward form, for reasons which will become apparent later.

We can extend the process further, to as many levels as we like. Because the expressions quickly become very cumbersome to write down in full, and a sheet of paper is only so big, we require some new notation.

Taking the *structure* of the continued fraction as a given,[4] we note that the only feature distinguishing different continued fractions is the sequence of *coefficients*, the numerators and denominators appearing in each of the nested fractions. So the first expression above could be more succinctly written[5] as

$$1 + \frac{1}{2+}\frac{1}{2}$$

and similarly, the second would be written as

$$1 + \frac{1}{2+}\frac{1}{2+}\frac{1}{2}$$

We can streamline this notation still further by restricting our attention to *simple* continued fractions,[6] for which all the numerators are equal to 1. Then all the relevant information is carried by the initial term (which we allow to be any integer) and the sequence of denominators (which we assume to be *positive* integers), and we write

$$a_0 + \frac{1}{a_1+}\frac{1}{a_2+}\cdots\frac{1}{a_n} = [a_0; a_1, a_2, \ldots, a_n]$$

Provided we do eventually stop after some finite number of steps, the resulting continued fraction can be simplified and written as a rational number in the usual way. Thus far, we have not really done anything new, but have simply replaced the usual method of writing rational numbers with another, rather more cumbersome, technique.

But what if we don't stop? What happens if we consider the expression

$$[a_0; a_1, a_2, \ldots]$$

where the sequence never terminates? Does this expression have any meaning? How does it relate to the finite continued fractions $[a_0; a_1, \ldots, a_n]$? As we shall see, the answers to these questions lead us into deeper waters.

4 To infinity, and... well, to infinity.

Perhaps the simplest example of an infinite continued fraction is the following:

$$[1; 1, 1, \ldots] = 1 + \frac{1}{1 + \frac{1}{1+\frac{1}{1+\cdots}}}$$

[4]We can define this structure recursively, as "a continued fraction X has the form $X = a + \frac{b}{Y}$, where a, b are integers and Y is again a continued fraction."

[5]Again, resist the urge to simplify it to $\frac{7}{5}$; it is not only the value of the fraction which is of interest, but also the structure, and so both must be preserved.

[6]Also called *standard* continued fractions, or continued fractions in canonical form.

Suppose this is to stand for some real number α. Then it must be the case that

$$\alpha = 1 + \frac{1}{\alpha}$$

from which we can quickly deduce that $\alpha = \frac{1+\sqrt{5}}{2}$, the golden ratio. But in what sense is this equal to the infinite continued fraction represented above? We will address this question in the next section after looking at one more example, which deals with the problem of computing a continued fraction which is to represent a specific α.

Suppose the continued fraction $[a_0; a_1, \ldots]$ is to stand for[7] the real number $\sqrt{2}$. The requirement that a_n be a positive integer for each $n \geq 1$ implies that a_0 must be the greatest integer less than or equal to $\sqrt{2}$, so $a_0 = \lfloor\sqrt{2}\rfloor = 1$, and we have

$$\sqrt{2} = \lfloor\sqrt{2}\rfloor + \{\sqrt{2}\} = 1 + (\sqrt{2} - 1)$$

where $\{x\}$ denotes the fractional part of x. Now to compute a_1, we note that

$$\frac{1}{a_1+}\frac{1}{a_2+}\cdots = \sqrt{2} - 1$$

and hence

$$a_1 + \frac{1}{a_2+}\cdots = \frac{1}{\sqrt{2}-1} = \sqrt{2} + 1$$

Thus as above, $a_1 = \lfloor\sqrt{2} + 1\rfloor = 2$, and we again have

$$\frac{1}{a_2+}\frac{1}{a_3+}\cdots = \{\sqrt{2} + 1\} = \sqrt{2} - 1$$

At this point it is clear that the pattern will continue to repeat, and so $a_n = 2$ for every $n \geq 1$. Indeed, if we write

$$\alpha = [1; 2, 2, 2, \ldots] = 1 + \frac{1}{2 + \frac{1}{2+\frac{1}{2+\cdots}}} = 1 + \frac{1}{\alpha + 1}$$

we immediately find the positive solution $\alpha = \sqrt{2}$.

This example illustrates the general algorithm for finding the continued fraction representation of a given number, rational or irrational. The integer part of the number gives the first coefficient, while the inverse of the fractional part is fed back into the algorithm to compute the second coefficient, and the process is repeated.[8]

However, we must now work out in what sense these expressions are equal to the numbers we claim they represent. Since we are dealing with infinite sequences of coefficients, there is bound to be a question of convergence lurking in the shadows, and it is to this matter which we now turn our attention.

[7] Whatever "stand for" means. At the moment, we have not proved anything rigorously, and these infinite sequences are to be dealt with simply as formal symbols.

[8] Prove that if our original number α is rational, then this process will terminate.

5 A matter of convergence

The trick used in the previous section to go from the continued fraction $[1; 1, \ldots]$ to the number $\alpha = \frac{1+\sqrt{5}}{2}$ relies on the fact that the sequence of coefficients is periodic. To deal with the general case, we must look at the sequence $\{x_n\}$ of *convergents* given by

$$x_n = [a_0; a_1, \ldots, a_n] = \frac{p_n}{q_n}$$

This is just a finite continued fraction of the form we have already dealt with; it is apparent that the convergents x_n are all rational, and the next step is to find formulae for p_n and q_n in terms of the coefficients a_n.

For $\alpha = \sqrt{2}$, we found the continued fraction expansion $[1; 2, 2, 2, \ldots]$. A little computation, which the reader is encouraged to work through, shows that the sequence of convergents is given by

$$(x_0, x_1, x_2, x_3, x_4, \ldots) = \left(\frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}, \ldots\right)$$

Note that squaring each of these terms gives

$$(x_0^2, x_1^2, x_2^2, x_3^2, x_4^2, \ldots) = \left(\frac{1}{1}, \frac{9}{4}, \frac{49}{25}, \frac{289}{144}, \frac{1681}{841}, \ldots\right)$$

which makes it plausible, at least, that $x_n \to \sqrt{2}$ as n goes to infinity.[9]

In the case of $\alpha = \frac{1+\sqrt{5}}{2}$, we have the expansion $[1; 1, 1, \ldots]$, and the reader is again encouraged to verify that

$$(x_0, x_1, x_2, x_3, x_4, \ldots) = \left(\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \ldots\right)$$

We immediately recognise our old friend the Fibonacci sequence in both the numerators and the denominators, suggesting that for this particular continued fraction, the value of p_n is given by the recursion $p_n = p_{n-1} + p_{n-2}$, and similarly for q_n, with initial conditions $p_{-1} = 1$, $q_{-1} = 0$, $p_0 = q_0 = 1$.

In general, of course, we must modify this formula so that p_n and q_n depend on a_n; upon examining the sequence of convergents for $\sqrt{2}$, we are led to conjecture the following rules, which we will then prove to be correct:

$$\begin{aligned} p_n &= a_n p_{n-1} + p_{n-2} \qquad p_{-1} = 1 \qquad p_0 = a_0 \\ q_n &= a_n q_{n-1} + q_{n-2} \qquad q_{-1} = 0 \qquad q_0 = 1 \end{aligned}$$

Taking this as our *definition* of p_n and q_n, we consider the function

$$f_n(t) = \frac{p_n t + p_{n-1}}{q_n t + q_{n-1}}$$

[9] In fact, each of these convergents $\frac{p}{q}$ satisfies Pell's equation $p^2 - 2q^2 = \pm 1$, and it can be shown that these are the only integral solutions.

where $t \in \mathbb{R}^+$, and show by induction that $f_n(t) = [a_0; a_1, \ldots, a_n, t]$.

The case $n = 0$ reduces to the statement that

$$a_0 + \frac{1}{t} = \frac{p_0 t + p_{-1}}{q_0 t + q_{-1}}$$

which is immediate, given the specified initial values. Now whenever the result holds for $n - 1$, we have

$$\begin{aligned}
[a_0; a_1, \ldots, a_n, t] &= \left[a_0; a_1, \ldots, a_{n-1}, a_n + \frac{1}{t}\right] \\
&= f_{n-1}\left(a_n + \frac{1}{t}\right) \\
&= \frac{p_{n-1}(a_n + \frac{1}{t}) + p_{n-2}}{q_{n-1}(a_n + \frac{1}{t}) + q_{n-2}} \\
&= \frac{p_n + \frac{1}{t} p_{n-1}}{q_n + \frac{1}{t} q_{n-1}} \\
&= \frac{p_n t + p_{n-1}}{q_n t + q_{n-1}} \\
&= f_n(t)
\end{aligned}$$

Taking the limit of both sides as $t \to \infty$ proves that $x_n = \frac{p_n}{q_n}$, so this recursion gives us the sequence of convergents, as claimed.

Now consider the 2×2 matrices A_n defined by

$$A_n = \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix}$$

Using the recursive definition of p_n and q_n given above, we have

$$\begin{aligned}
A_n &= \begin{pmatrix} a_n p_{n-1} + p_{n-2} & p_{n-1} \\ a_n q_{n-1} + q_{n-2} & q_{n-1} \end{pmatrix} \\
&= \begin{pmatrix} p_{n-1} & p_{n-2} \\ q_{n-1} & q_{n-2} \end{pmatrix} \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} p_0 & p_{-1} \\ q_0 & q_{-1} \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix}
\end{aligned}$$

This fact in and of itself does not make the computations significantly easier, but since each of the matrices in the product has determinant -1, it is now transparent that

$$p_n q_{n-1} - p_{n-1} q_n = (-1)^{n+1}$$

Hence the difference between two consecutive convergents is

$$x_n - x_{n-1} = \frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}} = \frac{p_n q_{n-1} - p_{n-1} q_n}{q_n q_{n-1}} = \frac{(-1)^{n+1}}{q_n q_{n-1}}$$

Now we can write x_n without any reference to p_n at all:

$$x_n = x_0 + \sum_{k=1}^{n} x_k - x_{k-1} = a_0 + \sum_{k=1}^{n} \frac{(-1)^{k+1}}{q_k q_{k-1}}$$

From the definition of q_n and the fact that $a_n \geq 1$ for all $n \geq 1$, it follows that (q_n) is a strictly increasing sequence, hence the alternating series in the above sum converges. This proves that $\alpha = \lim_{n\to\infty} x_n$ does in fact exist, and further establishes that

$$x_0 < x_2 < \cdots < x_{2n} < \cdots < \alpha < \cdots < x_{2n+1} < \cdots < x_3 < x_1$$

where the even convergents approach α monotonically from below; the odd convergents from above.

6 Rational approximations

Based on the nature of the convergence described in the previous section, we have the following error estimate:

$$|\alpha - x_n| < |x_{n+1} - x_n| = \frac{1}{q_{n+1}q_n}$$

In particular, since $q_{n+1} > q_n$, every convergent $x_n = \frac{p_n}{q_n}$ satisfies

$$\left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{q_n^2}$$

We can obtain a tighter result by considering two successive convergents x_n and x_{n+1}. Using the inequality $2xy \leq x^2 + y^2$, we see that

$$\frac{1}{q_{n+1}q_n} < \frac{1}{2q_{n+1}^2} + \frac{1}{2q_n^2}$$

where the inequality is strict since $q_{n+1} \neq q_n$, and hence

$$\left|\alpha - \frac{p_{n+1}}{q_{n+1}}\right| + \left|\alpha - \frac{p_n}{q_n}\right| = |x_{n+1} - x_n| < \frac{1}{2q_{n+1}^2} + \frac{1}{2q_n^2}$$

It follows that for one of the two convergents $\frac{p}{q}$, we have

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{2q^2}$$

Although we shall not prove this fact here,[10] it is worth noting that *any* rational number $\frac{p}{q}$ which satisfies this inequality must be found in the sequence of

[10]See Theorem 19 in Khinchin's book [4].

convergents to α. Not only does this sequence give us good rational approximations, there is a very definite sense in which it gives us *all* the good rational approximations!

For some particular values of n, it may happen that the convergent x_n is especially close to α. When does this happen? Using the recursive definition of q_{n+1}, we have

$$\frac{1}{q_{n+1}q_n} = \frac{1}{q_n(a_{n+1}q_n + q_{n-1})} < \frac{1}{a_{n+1}}\frac{1}{q_n^2}$$

Thus the best approximations will be those convergents x_n for which a_{n+1} takes a large value; in particular, if the continued fraction expansion of α has unbounded coefficients, then for any constant $C > 0$, there exists a rational approximation satisfying

$$\left|\alpha - \frac{p}{q}\right| < \frac{C}{q^2}$$

A little grunt work[11] shows that given a fixed N, the set of real numbers whose continued fraction expansions have coefficients bounded by N is a Cantor set of measure zero. It follows that for almost every real number α and for every $C > 0$, we can find an rational number satisfying the above approximation.

This stands in stark contrast to the situation concerning the set of Diophantine numbers, that is, those real numbers α such that for some $C > 0$, $\delta > 0$, any rational number $\frac{p}{q} \neq \alpha$ satisfies

$$\left|\alpha - \frac{p}{q}\right| \geq \frac{C}{q^{2+\delta}}$$

This set can be shown to have full Lebesgue measure on $\mathbb{R}$; in other words, for almost every real number α, we can find rational approximations to the order of the inverse square of the denominator, but no better.

One particular approximation is worth noting. The best-known rational approximation to π is the fraction $\frac{22}{7}$, which first differs from the true value of the constant in the third decimal place. The first few coefficients of the continued fraction expansion for π are given by

$$\pi = [3; 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, \ldots]$$

So $\frac{22}{7}$ is, in fact, $x_1 = [3; 7]$, and has $a_2 = 15$, which results in a reasonable approximation to π. Given the startling magnitude of $a_4 = 292$, an even better approximation is $x_3 = [3; 7, 15, 1] = \frac{355}{113}$. Indeed, if we write out the first few digits of the decimal expansions, we see that

$$\begin{aligned} \frac{22}{7} &= 3.14285714\ldots \\ \frac{355}{113} &= 3.14159292\ldots \\ \pi &= 3.14159265\ldots \end{aligned}$$

[11] In customary fashion, this is omitted here and left for the reader to carry out.

This approximation gives us π correct to more than twice as many digits as our previous one! No other approximation to π with any sort of reasonable denominator is anywhere near this accurate, and we would have been hard pressed to stumble across this one by random chance, without the theory of continued fractions to guide us.

7 Patterns

When we represent real numbers by their decimal expansions, we can separate them into three categories, according to whether the decimal expansion terminates, becomes eventually periodic, or does neither. If we adopt the convention that every decimal expansion is infinite (adding an infinite string of zeroes if necessary), then the first category becomes part of the second; in either case, these two categories represent rational numbers, while the third category encompasses the irrationals.

As a general rule, the decimal expansion of an irrational number does not exhibit any noticeable patterns; this is not true of continued fraction expansions. We have already seen that the rationals are precisely those real numbers whose continued fraction expansion is finite. The natural question to ask next is, what happens if the expansion is periodic or eventually periodic? In this case, we have the following:

$$\begin{aligned}
\alpha &= [a_0; a_1, \ldots, a_n, \overline{a_{n+1}, \ldots, a_{n+k}}] \\
\beta &= [\overline{a_{n+1}; a_{n+2}, \ldots, a_{n+k}}] \\
\alpha &= f_n(\beta) = \frac{p_n\beta + p_{n-1}}{q_n\beta + q_{n-1}} \\
&= f_{n+k}(\beta) = \frac{p_{n+k}\beta + p_{n+k-1}}{q_{n+k}\beta + q_{n+k-1}}
\end{aligned}$$

It follows that β is the root of a quadratic equation with integer coefficients, so $\beta = p + q\sqrt{r}$ for some $p, q, r \in \mathbb{Q}$, and hence α has the same form.

It is not too hard to show that the converse of this statement is true as well; any real number which can be written in the form given above will have a continued fraction expansion which is eventually periodic.

This is not the whole story, though—even certain transcendental numbers have highly regular continued fraction expansions. For example, we have

$$\begin{aligned}
e &= [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, \ldots] \\
\tan(1) &= [1; 1, 1, 3, 1, 5, 1, 7, 1, 9, 1, 11, \ldots]
\end{aligned}$$

and similar (though more complicated) patterns hold for $e^{l/m}$, $\tan(1/n)$, etc. Earlier, we saw the expansion for π, which does not exhibit any noticeable pattern. However, if we broaden our horizons beyond simple continued fractions, and allow numerators of different values, then we have such equations as

$$\frac{4}{\pi} = 1 + \frac{1}{2+}\frac{9}{2+}\frac{25}{2+}\frac{49}{2+}\cdots$$

and many more besides, which we will not investigate further here. Rather, the interested reader is directed to the references[12] for proofs of these remarkable representations, and for further explorations in this area.

One final comment is in order. There are many unexpected statistical and ergodic properties of continued fractions which we have not mentioned here; for example, almost every real number α has the property that the geometric mean of the first n coefficients in its continued fraction expansion converges to a number, independent of α, known as Khinchin's constant. This and other magical-seeming results are discussed in section 4.8 of [1].

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[12]The formula concerning π can be found in the appendix of [5], and the results concerning e and the tangent function are proved in [3] and [7].