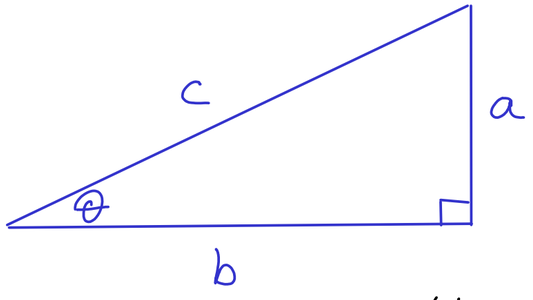
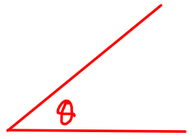
How do we define the sine function?

① Ratio of triangle sides

$$\sin\theta = \frac{a}{c}$$



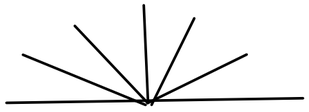
- What if θ is <0? or $>90°$?
- How do we define "angle"?



If I claim that $\theta = 30°$, how do you verify it?

Six copies should lead to a straight line ....

$(6 \times 30° = 180°)$



What if I claim that $\theta = \sqrt{2}°$?

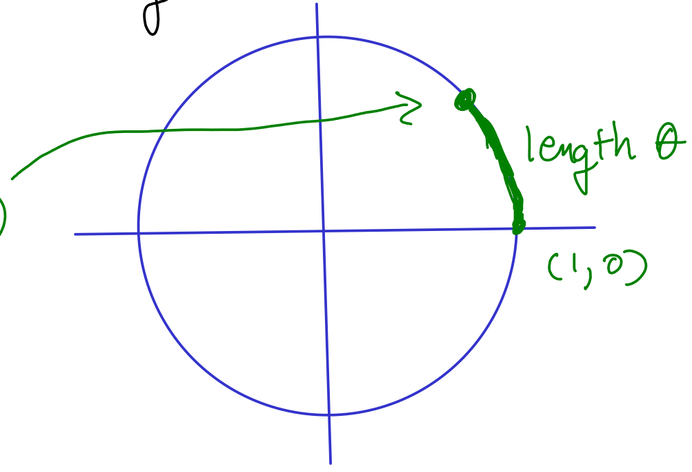
$1.4 < \sqrt{2} < 1.5$ etc. (add more digits)

$\sqrt{2}$ is irrational! (requires pf)

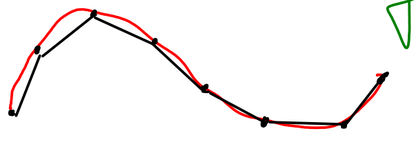
② Point on unit circle

coords $= (\cos\theta, \sin\theta)$

(use radians)

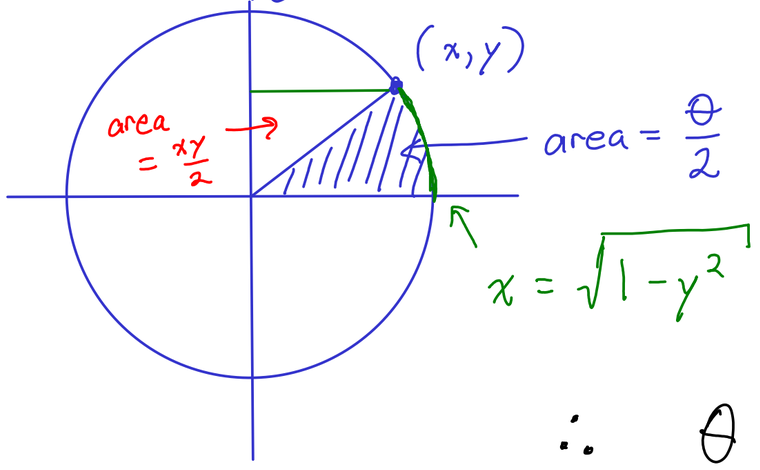


- How do we define "length"?



approx. by line segments

③ Via inverse function arcsin using area....



Area of triangle ∪ sector

$= \frac{y}{2}\sqrt{1-y^2} + \frac{\theta}{2}$ (basic geometry)

$= \int_0^y \sqrt{1-t^2}\,dt$ (calculus)

(need to define integration)

$\therefore \quad \theta = -y\sqrt{1-y^2} + 2\int_0^y \sqrt{1-t^2}\,dt$

Defines $\theta = \arcsin y$ via calculus.... then invert to get sin....

④ Simple harmonic motion / differential equation

(need to define derivatives)

$y = \sin x$ is the unique solution of $y'' + y = 0$, $y(0)=0$, $y'(0)=1$

(requires DE theory ...)

Defining an object via its properties!

Already did this with $\sqrt{2}$:
the unique real number x
such that $x^2 = 2$ & $x > 0$.

(5) $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

(another axiomatic approach)

sin & cos are the unique functions satisfying these identities (& a bit more)

(6) Euler's formula:

What is e^{ix}? Is this formula a definition or a theorem?

$$\left.\begin{aligned} e^{ix} &= \cos x + i\sin x \\ e^{-ix} &= \cos x - i\sin x \end{aligned}\right\} \Rightarrow \sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

(e.g. define e^{ix} using power series)

(7) Power series:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots$$

(need to make sense of infinite sums...)