Multifractal analysis via thermodynamics

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Outline

1. Motivation and definitions
   - A multifractal decomposition
   - Carathéodory dimension characteristics and fine spectra
   - Capacity entropy and coarse spectra
   - Pressure and variational principles

2. Results: old and new
   - Known results: thermodynamics vs. saturation
   - New results using thermodynamics
   - Other multifractal spectra

Birkhoff averages

General setting:

- X is a compact metric space;
- $f: X \to X$ is a continuous map;
- $\Phi = (\varphi_1, \dots, \varphi_d): X \to \mathbb{R}^d$ is a continuous function.

Birkhoff sums: $S_n\varphi_i(x) = \sum_{k=0}^{n-1} \varphi_i(f^k(x))$.

QUESTION: What is the asymptotic behaviour of $\frac{1}{n}S_n\Phi(x) \in \mathbb{R}^d$?

- No invariant measure is specified, so can't use ergodic theorem.
- When we do have an invariant measure, orbits with atypical behaviour still influence finite-time behaviour of typical orbits.

Level sets

The level sets for Birkhoff averages are

$$K(\boldsymbol{\alpha}) = \left\{ x \in X \,\middle|\, \frac{1}{n} S_n \Phi(x) \to \boldsymbol{\alpha} \right\}, \qquad \boldsymbol{\alpha} \in \mathbb{R}^d$$

QUESTION: What are these sets like? How big are they?

We want to examine the multifractal decomposition

$$X = \left(\bigcup_{\boldsymbol{\alpha} \in \mathbb{R}^d} K(\boldsymbol{\alpha}) \right) \cup \hat{X}.$$

Organises trajectories by asymptotic statistical properties.

$I(\Phi) = \{ \int \Phi \, d\mu \mid \mu \in \mathcal{M}^f(X) \} \subset \mathbb{R}^d$. Only consider $\boldsymbol{\alpha} \in I(\Phi)$.

Full multifractal decomposition

Given $x \in X$, let

$$A(x) = \left\{ \text{accumulation points of } \frac{1}{n} S_n \Phi(x) \right\}.$$

This is a compact connected subset of $\mathbb{R}^d$.

$$\mathcal{G} = \{E \subset \mathbb{R}^d \mid E \text{ is compact and connected}\}$$

The level set associated to $E \in \mathcal{G}$ is

$$K(E) = \{x \in X \mid A(x) = E\}.$$

This gives a complete multifractal decomposition

$$X = \bigcup_{E \in \mathcal{G}} K(E).$$

A symbolic example

Example

Full shift on two symbols.

- $X = \Sigma_2^+ = \{0,1\}^{\mathbb{N}}$.
- $f = \sigma$, the shift map.
- $\varphi = \mathbf{1}_{[0]}$, the characteristic function of the 1-cylinder [0].

Then $\frac{1}{n} S_n \varphi(x)$ is the frequency of 0s in the first n symbols.

Observations

- $K(\alpha)$ is dense for $0 \leq \alpha \leq 1$, and empty otherwise.
- Level sets are f-invariant but not compact.

It feels like $K(0)$ is smaller, more restrictive, than $K(\frac{1}{2})$. But how?

Hausdorff dimension

X a separable metric space. For $Z \subset X$, let

$$\mathcal{D}(Z,\varepsilon) = \left\{ \{(x_i, r_i)\} \mid x_i \in Z, r_i \leq \varepsilon, Z \subset \bigcup_i B(x_i, r_i) \right\}.$$

The s-dimensional Hausdorff outer measure of $Z \subset X$ is

$$m_H(Z, s) = \lim_{\varepsilon \to 0} \inf_{\mathcal{D}(Z,\varepsilon)} \sum_i r_i^s.$$

The Hausdorff dimension of $Z \subset X$ is

$$\begin{aligned} \dim_H(Z) &= \sup\{s \geq 0 \mid m_H(Z, s) = +\infty\} \\ &= \inf\{s \geq 0 \mid m_H(Z, s) = 0\}. \end{aligned}$$

Topological entropy

The Bowen ball of order n and radius δ is

$$B(x,n,\delta)=\{y\in X \mid d(f^k(x),f^k(y))<\delta \text{ for all } 0\le k\le n\}.$$

Using these, we replace $\mathcal{D}(Z,\varepsilon)$ with

$$\mathcal{P}(Z,N,\delta)=\Big\{\{(x_i,n_i)\} \mid x_i\in Z, n_i\ge N, Z\subset \bigcup B(x_i,n_i,\delta)\Big\}.$$

The s-dimensional entropy outer measure at scale δ of $Z\subset X$ is

$$m_h(Z,s,\delta)=\lim_{N\to\infty}\inf_{\mathcal{P}(Z,N,\delta)}\sum_i e^{-n_i s}.$$

The topological entropy (in the sense of Bowen) of $Z\subset X$ is

$$\begin{aligned} h_{\text{top}}(Z,\delta) &= \sup\{s\in\mathbb{R} \mid m_h(Z,s,\delta)=+\infty\} \\ &= \inf\{s\in\mathbb{R} \mid m_h(Z,s,\delta)=0\}, \\ h_{\text{top}}(Z) &= \lim_{\delta\to 0} h_{\text{top}}(Z,\delta). \end{aligned}$$

The entropy spectrum for Birkhoff averages

We treat entropy as a dimensional quantity:

- h_{top} characterises subsets of X, not just global dynamics.

The entropy spectrum for Birkhoff averages of Φ is the function $\mathcal{B} \colon \mathbb{R}^d \mapsto [0, \infty) \cup \{-\infty\}$ given by

$$\mathcal{B}(\alpha) = h_{\text{top}} K(\alpha).$$

More generally, $\mathcal{B} \colon \mathcal{G} \mapsto [0, \infty) \cup \{-\infty\}$ is given by

$$\mathcal{B}(E) = h_{\text{top}} K(E).$$

The multifractal miracle

The definition of $\mathcal{B}(\alpha)$ is roundabout, relying on

1. asymptotically defined function ($\lim \frac{1}{n} S_n \Phi$)—only measurable;
2. one-parameter family of outer measures $m_h(\cdot, s, \delta)$;
3. critical value of s for each set $K(\alpha)$.

Why should we get a "reasonable" function?

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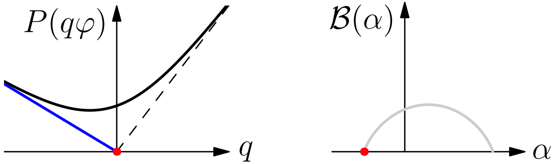
KEY IDEA: $\mathcal{B}(\alpha)$ is (often) the Legendre transform of the topological pressure function: $\mathcal{B}(\alpha) = \inf_{\mathbf{q} \in \mathbb{R}^d} (P(\langle \mathbf{q}, \Phi \rangle) - \langle \mathbf{q}, \alpha \rangle)$.



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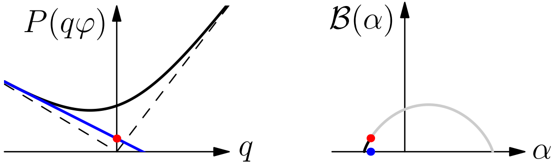

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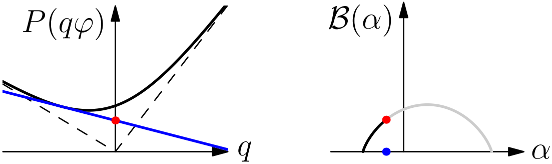

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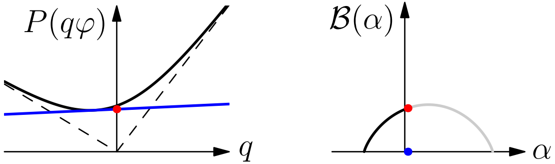

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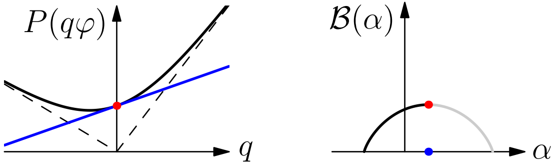

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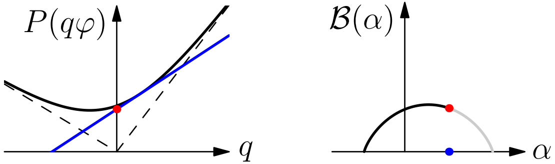

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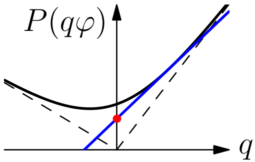

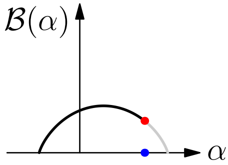

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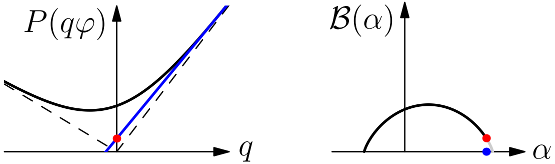

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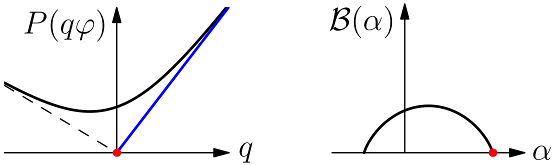

Capacity entropy

Definition of entropy used covers $\{B(x_i, n_i, \delta)\}$ with $n_i \geq N$. Replacing this with $n_i = N$ yields the capacity entropies.

$$\mathcal{P}'(Z, N, \delta) = \left\{\{x_i\} \subset Z \mid Z \subset \bigcup B(x_i, N, \delta)\right\}$$

$$\Lambda_N(Z, \delta) = \inf\{\#E \mid E \in \mathcal{P}'(Z, N, \delta)\}$$

$$\underline{h}_{\mathrm{top}} Z = \lim_{\delta \to 0} \varliminf_{N \to \infty} \frac{1}{N} \log \Lambda_N(Z, \delta)$$

$$\overline{h}_{\mathrm{top}} Z = \lim_{\delta \to 0} \varlimsup_{N \to \infty} \frac{1}{N} \log \Lambda_N(Z, \delta)$$

- $h_{\mathrm{top}}\, Z \leq \underline{h}_{\mathrm{top}} Z \leq \overline{h}_{\mathrm{top}} Z$.
- If Z is compact and invariant then all three are equal.
- If Z is dense in X then $\underline{h}_{\mathrm{top}} Z = \overline{h}_{\mathrm{top}} Z = h_{\mathrm{top}}(X, f)$.

Another version of capacity entropy

- $h_{\text{top}}\, Z \leq \underline{h}_{\text{top}} Z \leq \overline{h}_{\text{top}} Z$, but the inequalities are often strict.

Many interesting sets look like $Z = \bigcup_N \bigcap_{n \geq N} Z_n$. Let

$$\underline{h}_{\text{top}}\{Z_n\} = \lim_{\delta \to 0} \varliminf_{n \to \infty} \frac{1}{n} \log \Lambda_n(Z_n, \delta),$$

$$\overline{h}_{\text{top}}\{Z_n\} = \lim_{\delta \to 0} \varlimsup_{n \to \infty} \frac{1}{n} \log \Lambda_n(Z_n, \delta).$$

Then for $Z = \bigcup_N \bigcap_{n \geq N} Z_n$, we have

- $h_{\text{top}}\, Z \leq \underline{h}_{\text{top}}\{Z_n\} \leq \overline{h}_{\text{top}}\{Z_n\}$.

Coarse multifractal spectra

Given $U \ni \alpha$ open, let

$$G_n(U) = \left\{ x \in X \mid \frac{1}{n} S_n \Phi(x) \in U \right\}.$$

Then we have

$$K(\alpha) = \bigcap_{U \ni \alpha} \bigcup_{N \geq 1} \bigcap_{n \geq N} G_n(U).$$

The coarse multifractal spectra for Φ are

$$\underline{\mathcal{B}}(\alpha) = \inf_{U \ni \alpha} \underline{h}_{\text{top}} \{G_n(U)\},$$
$$\overline{\mathcal{B}}(\alpha) = \inf_{U \ni \alpha} \overline{h}_{\text{top}} \{G_n(U)\}.$$

We expect to find that

$$\mathcal{B}(\alpha) = \underline{\mathcal{B}}(\alpha) = \overline{\mathcal{B}}(\alpha).$$

Topological pressure

- Bowen balls correspond to orbit segments.
- Entropy treats all orbit segments of the same length equally.
- Pressure assigns weights based on a potential function ξ.

$$\Lambda_N(X,\xi,\delta) = \inf\left\{ \sum e^{S_N\xi(x_i)} \mid \{x_i\} \in \mathcal{P}'(X,N,\delta)\right\}$$

$$P(\xi) = \lim_{\delta\to 0} \varlimsup_{N\to\infty} \frac{1}{N} \log \Lambda_N(X,\xi,\delta)$$

When $\phi = 0$ we get entropy: $h_{\text{top}}(X,f) = P(0)$.

FACT: $P\colon C(X) \to [-\infty,\infty]$ is continuous and convex.

Variational principles

The pressure function satisfies

$$P(\xi) = \sup\left\{h_\mu(f) + \int \xi \, d\mu \mid \mu \in \mathcal{M}^f(X)\right\}.$$

In particular, $h_{\text{top}}(X, f) = \sup\{h_\mu(f) \mid \mu \in \mathcal{M}^f(X)\}$. What about non-compact sets? It is not true in general that

$$h_{\text{top}} Z = \sup\{h_\mu(f) \mid \mu \in \mathcal{M}^f(Z)\}.$$

However, we expect the conditional variational principle

$$\mathcal{B}(\alpha) = h_{\text{top}} K(\alpha) = \sup\{h_\mu(f) \mid \mu \in \mathcal{M}^f(K(\alpha))\}.$$

Similarly, it is true for many systems that

$$\mathcal{B}(\alpha) = h_{\text{top}} K(\alpha) = \sup\left\{h_\mu(f) \mid \mu \in \mathcal{M}^f(X), \int \Phi \, d\mu = \alpha\right\}.$$

Justification of Legendre transform

The variational principle justifies the Legendre transform. Recall that $\mathcal{M}^f(X) \subset C(X)^*$ with the pairing $\langle \mu, \xi \rangle = \int \xi \, d\mu$.

$$P(\xi) = \sup_\mu (h_\mu(f) + \langle \mu, \xi \rangle)$$

says that $P(\cdot)\colon C(X) \to \mathbb{R}$ is the Legendre transform of $h_\cdot(f)\colon C(X)^* \to \mathbb{R}$. Legendre transform duality implies that

$$h_\mu(f) = \inf_\xi (P(\xi) - \langle \mu, \xi \rangle),$$

provided $\mu \mapsto h_\mu(f)$ is upper semi-continuous. Then if CVP holds and infimum is achieved for $\xi \in \operatorname{span}\{\varphi_1, \dots, \varphi_d\}$,

$$\begin{aligned} h_{\mathrm{top}} K(\boldsymbol{\alpha}) &= \sup\{h_\mu(f) \mid \langle \mu, \Phi \rangle = \boldsymbol{\alpha}\} \\ &= \inf_{\mathbf{q}} (P(\langle \mathbf{q}, \Phi \rangle) - \langle \mathbf{q}, \boldsymbol{\alpha} \rangle). \end{aligned}$$

Summary of expected results

We expect all of the following quantities to be equal.

- Fine multifractal spectrum $\mathcal{B}(\boldsymbol{\alpha}) = h_{\text{top}}\, K(\boldsymbol{\alpha})$
- Coarse multifractal spectra $\underline{\mathcal{B}}(\boldsymbol{\alpha})$ and $\overline{\mathcal{B}}(\boldsymbol{\alpha})$
- Legendre transform $\inf\{P(\langle \mathbf{q}, \Phi\rangle) - \langle \mathbf{q}, \boldsymbol{\alpha}\rangle) \mid \mathbf{q} \in \mathbb{R}^d\}$
- $\sup\{h_\mu(f) \mid \mu \in \mathcal{M}^f(X), \int \Phi\, d\mu = \boldsymbol{\alpha}\}$
- $\sup\{h_\mu(f) \mid \mu \in \mathcal{M}^f(K(\boldsymbol{\alpha}))\}$

Question: What conditions do we need to impose on the system f and the observables φ_i for these all to coincide?

Nature of known results

- Large literature on the subject, very many examples studied, but relatively few axiomatic approaches.
- Many variations on “multifractal spectrum”: different local quantities, different global quantities.

Most results come from one of two methods:

THERMODYNAMIC APPROACH. Obtain invariant measures supported on $K(\alpha)$ as equilibrium states for suitable potential functions.

SATURATION APPROACH. Use some version of specification property to build non-invariant measures supported on $K(\alpha)$.

Known results with thermodynamics

The most general thermodynamic results rely on existence of unique equilibrium states.

Theorem (Barreira–Saussol–Schmeling (2002))

X a compact metric space, $f : X \to X$ continuous. Suppose

- *entropy map $\mu \mapsto h_\mu(f)$ upper semi-continuous, and*
- *every $\xi \in \operatorname{span}\{\varphi_1, \dots, \varphi_d\}$ has a unique equilibrium state.*

Then for every $\alpha \in \operatorname{int} I(\Phi)$, we have

$$\begin{aligned} h_{\text{top}} K(\alpha) &= \max\left\{ h_\mu(f) \mid \mu \in \mathcal{M}^f(K(\alpha)) \right\} \\ &= \max\left\{ h_\mu(f) \mid \mu \in \mathcal{M}^f(X), \int \Phi \, d\mu = \alpha \right\} \\ &= \inf\{ P(\langle \mathbf{q}, \Phi \rangle) - \langle \mathbf{q}, \alpha \rangle \mid \mathbf{q} \in \mathbb{R}^d \}. \end{aligned}$$

Examples for thermodynamic approach

Upper semi-continuity of entropy holds if f is entropy-expansive.

Uniqueness of equilibrium state is more delicate.

- f expansive with specification $\Rightarrow$ uniqueness for all Bowen potentials.
- f uniformly hyperbolic $\Rightarrow$ Hölder continuous potentials are Bowen, and f is expansive with specification.

Important examples where uniqueness does not hold on $\mathrm{span}\{\varphi\}$.

Example

Let $f\colon [0,1] \to [0,1]$ be the Manneville–Pomeau map $f(x) = x + x^{1+\varepsilon} \pmod 1$, and let $\varphi(x) = -\log|f'(x)|$ be the geometric potential. Then $q \mapsto P(q\varphi)$ is non-differentiable at $q = 1$.

Known results with saturation

Saturation results rely on a version of the specification property.

Theorem (Takens–Verbitskiy (2003), Olsen (2003), Pfister–Sullivan (2007))

X a compact metric space, $f : X \to X$ continuous. Suppose

- *f has the uniform separation property, and*
- *f has the g-almost product property.*

Let $\Phi \in C(X)^d$ and $E \subset I(\Phi)$ closed and connected. Then

$$h_{\text{top}} K(E) = \inf_{\alpha \in E} \sup \left\{ h_\mu(f) \mid \mu \in \mathcal{M}^f(X), \int \Phi \, d\mu = \alpha \right\}$$
$$= \inf\{P(\langle \mathbf{q}, \Phi \rangle) - \langle \mathbf{q}, \alpha \rangle \mid \mathbf{q} \in \mathbb{R}^d, \alpha \in E\}.$$

Remark: Hypotheses satisfied for any entropy-expansive system with specification.

Differences between thermodynamics and saturation

Each approach has advantages and disadvantages.

Thermodynamic approach:

1. Gives CVP with $\sup\{h_\mu(f) \mid \mu \in \mathcal{M}^f(K(\alpha))\}$.
2. Only works for restricted class of functions.
3. Cannot deal with $\alpha \in \partial I(\Phi)$ or irregular sets $K(E)$.

Saturation appproach:

1. Gives no information about measures $\mu \in \mathcal{M}^f(K(\alpha))$.
2. Works for any continuous functions φ_i.
3. Gives results on $\partial I(\Phi)$ and on irregular sets $K(E)$.

GOAL: Using an idea due to Hofbauer, give conditions under which thermodynamic approach works for all continuous functions.

General results

Some relationships hold in complete generality.

Theorem (C. (2011))

X a compact metric space, $f: X \to X$ continuous, $\Phi \in C(X)^d$. Then for $\alpha \in \operatorname{int} I(\Phi)$ and $\xi \in C(X)$ we have

$$\begin{aligned} &\sup\{h_\mu(f) \mid \mu \in \mathcal{M}^f(K(\alpha))\} \\ &\quad \leq \mathcal{B}(\alpha) \leq \underline{\mathcal{B}}(\alpha) \leq \overline{\mathcal{B}}(\alpha) \\ &\quad \leq \sup\left\{h_\mu(f) \mid \mu \in \mathcal{M}^f(X), \int \Phi \, d\mu = \alpha\right\} \\ &\quad = \inf\{P(\langle \mathbf{q}, \Phi\rangle) - \langle \mathbf{q}, \alpha\rangle \mid \mathbf{q} \in \mathbb{R}^d\}. \end{aligned}$$

Equality can fail because measures in final supremum need not be ergodic, and hence need not be supported on $K(\alpha)$.

Comparison of approaches

Common theme in dimension theory: upper bounds on $\dim K(\alpha)$ are easy, lower bounds require a measure with $\mu(K(\alpha)) > 0$.

Saturation approach:

1. Measures with $\int \Phi \, d\mu = \alpha$ may not have generic points, but their ergodic components do.
2. Use the specification property to "glue together" orbits of generic points for these ergodic components and construct a non-invariant measure supported on $K(\alpha)$.

Thermodynamic approach:

1. Give conditions under which it suffices to consider ergodic measures in the supremum.
2. Ergodic μ with $\int \Phi \, d\mu = \alpha$ must be supported on $K(\alpha)$.

Idea of thermodynamic approach

THERMODYNAMIC APPROACH: To establish equality, produce a measure $\mu = \mu_{\boldsymbol{\alpha}} \in \mathcal{M}^f(K(\boldsymbol{\alpha}))$ and $\mathbf{q} \in \mathbb{R}^d$ with

$$h_\mu(f) = P(\langle \mathbf{q}, \Phi \rangle) - \langle \mathbf{q}, \boldsymbol{\alpha} \rangle.$$

In other words, μ is an equilibrium state for $\langle \mathbf{q}, \Phi \rangle$.

Theorem (Ruelle)

If $\mathbf{q} \mapsto P(\langle \mathbf{q}, \Phi \rangle)$ is differentiable at $\mathbf{q}$ and $\mu_{\mathbf{q}} \in \mathcal{M}^f(X)$ is an equilibrium state, then $\int \Phi \, d\mu_{\mathbf{q}} = \nabla_{\mathbf{q}} P(\langle \mathbf{q}, \Phi \rangle)$.

So if P is differentiable on $\text{span}\{\varphi_1, \dots, \varphi_d\}$ and equilibrium states exist, then we're nearly home.

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In other words, μ is an equilibrium state for $\langle \mathbf{q}, \Phi \rangle$.

Theorem (Ruelle)

If $\mathbf{q} \mapsto P(\langle \mathbf{q}, \Phi \rangle)$ is differentiable at $\mathbf{q}$ and $\mu_{\mathbf{q}} \in \mathcal{M}^f(X)$ is an equilibrium state, then $\int \Phi \, d\mu_{\mathbf{q}} = \nabla_{\mathbf{q}} P(\langle \mathbf{q}, \Phi \rangle)$.

So if P is differentiable on $\operatorname{span}\{\varphi_1, \dots, \varphi_d\}$ and equilibrium states exist, then we're nearly home. The rest follows by showing

$$\{\nabla_{\mathbf{q}} P(\langle \mathbf{q}, \Phi \rangle) \mid \mathbf{q} \in \mathbb{R}^d\} \supset \operatorname{int} I(\Phi).$$

General continuous functions

What about Φ such that P is not differentiable?

Theorem (C. (2011))

X a compact metric space, $f : X \to X$ continuous. Suppose

1. *entropy map $\mu \mapsto h_\mu(f)$ upper semi-continuous, and*
2. *there is a dense subspace $D \subset C(X)$ such that every $\phi \in D$ has a unique equilibrium state.*

For any $\Phi \in C(X)^d$ and $\boldsymbol{\alpha} \in \operatorname{int} I(\Phi)$ we have

$$\begin{aligned}
&\sup\{h_\mu(f) \mid \mu \in \mathcal{M}^f(K(\boldsymbol{\alpha}))\} \\
&\quad = \mathcal{B}(\boldsymbol{\alpha}) = \underline{\mathcal{B}}(\boldsymbol{\alpha}) = \overline{\mathcal{B}}(\boldsymbol{\alpha}) \\
&\quad = \sup\left\{h_\mu(f) \mid \mu \in \mathcal{M}^f(X), \int \Phi \, d\mu = \boldsymbol{\alpha}\right\} \\
&\quad = \inf\{P(\langle \mathbf{q}, \Phi\rangle) - \langle \mathbf{q}, \boldsymbol{\alpha}\rangle \mid \mathbf{q} \in \mathbb{R}^d\}.
\end{aligned}$$

Examples

Example (Bowen (1974))

If f is expansive with specification, then Bowen potentials are a dense subspace in $C(X)$, and every Bowen potential has a unique equilibrium state.

Example (Walters (1978), C.–Thompson (2011))

The β-shifts have a weaker version of the specification property that still guarantees uniqueness of the equilibrium state for a dense subspace of potentials.

REMARK: The version of specification guaranteeing uniqueness of equilibrium states for Bowen potentials is different from the version used by Pfister and Sullivan in their saturation results. It is not known whether their version implies uniqueness for a dense subspace of potentials.

Other local quantities

When f is conformal and $\varphi = \log |f'(x)|$, we get a decomposition into level sets for Lyapunov exponents.

If μ is a weak Gibbs measure for φ, then we get a decomposition into level sets for local entropies of μ.

The main results also go through if we consider $\varphi, \psi \in C(X)$ and define $K(\alpha) = \{x \mid \frac{S_n\varphi(x)}{S_n\psi(x)} \to \alpha\}$. Then if μ is weak Gibbs for φ and $\psi = \log |f'(x)|$, we get a decomposition into level sets for pointwise dimensions of μ.

Other dimensions

Hausdorff dimension is often of interest. More generally, given $u \in C(X)$, $u > 0$, Barreira and Schmeling define the u-dimension, and we can study $\dim_u K(\alpha)$.

$$\begin{aligned} u = 0 \quad &\Rightarrow \quad \dim_u = h_{\text{top}} \\ u = \log|f'| \quad &\Rightarrow \quad \dim_u = \dim_H \end{aligned}$$

Theorem (Bowen (1979), Barreira–Schmeling (2000), C. (2011))

$\dim_u Z = \inf\{t \geq 0 \mid P_Z(-tu) \leq 0\}$.

Need to make sense of $P_Z(\xi)$ for $Z \subset X$ non-compact.

Pressure for non-compact sets

Fix $Z \subset X$ and $\xi \in C(X)$. For each $s \in \mathbb{R}$, let

$$m_P(Z, \xi, s, \delta) = \lim_{N \to \infty} \inf_{\mathcal{P}(Z,N,\delta)} \sum_i e^{-n_i s + S_{n_i}\xi(x_i)}.$$

The topological pressure (in the sense of Pesin and Pitskel') is

$$\begin{aligned} P_Z(\xi, \delta) &= \sup\{s \in \mathbb{R} \mid m_P(Z, \xi, s, \delta) = +\infty\} \\ &= \inf\{s \in \mathbb{R} \mid m_P(Z, \xi, s, \delta) = 0\}, \\ P_Z(\xi) &= \lim_{\delta \to 0} P_Z(\xi, \delta). \end{aligned}$$

Pressure spectra

We can define a multifractal spectrum using pressure. Fix $\Phi \in C(X, \mathbb{R}^d)$. For each $\boldsymbol{\alpha} \in \mathbb{R}^d$ we get a pressure function

$$P_{K(\boldsymbol{\alpha})} \colon C(X) \to [-\infty, \infty].$$

Theorem (C. (2011))

X a compact metric space, $f : X \to X$ continuous. For any $\Phi \in C(X)^d$, $\xi \in C(X)$, and $\boldsymbol{\alpha} \in \operatorname{int} I(\Phi)$ we have

$$\begin{aligned}
&\sup \left\{ h_\mu(f) + \int \xi \, d\mu \mid \mu \in \mathcal{M}^f(K(\boldsymbol{\alpha})) \right\} \\
&\quad \le P_{K(\boldsymbol{\alpha})}(\xi) \le \inf_{U \ni \boldsymbol{\alpha}} \underline{P}_{\{G_n(U)\}}(\xi) \le \inf_{U \ni \boldsymbol{\alpha}} \overline{P}_{\{G_n(U)\}}(\xi) \\
&\quad \le \sup \left\{ h_\mu(f) + \int \xi \, d\mu \mid \mu \in \mathcal{M}^f(X), \int \Phi \, d\mu = \boldsymbol{\alpha} \right\} \\
&\quad = \inf \{ P(\langle \mathbf{q}, \Phi \rangle + \xi) - \langle \mathbf{q}, \boldsymbol{\alpha} \rangle \mid \mathbf{q} \in \mathbb{R}^d \}.
\end{aligned}$$

Main result for pressure spectra

Theorem (C. (2011))

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&\quad = \inf\{P(\langle \mathbf{q}, \Phi\rangle + \xi) - \langle \mathbf{q}, \boldsymbol{\alpha}\rangle \mid \mathbf{q} \in \mathbb{R}^d\}.
\end{aligned}$$

Idea of proof

Usual thermodynamic proof:

1. $T : q \mapsto P(q\varphi)$ differentiable
2. $I(\varphi) = [T'(-\infty), T'(+\infty)]$
3. μ_q an ergodic equilibrium state for $q\varphi$
4. $\alpha = T'(q) \Rightarrow \int \varphi \, d\mu_q = \alpha \Rightarrow \mu(K(\alpha)) = 1$

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Modified thermodynamic proof:

1. $\hat{\varphi} \approx \varphi$ such that $\hat{T} : q \mapsto P(q\hat{\varphi})$ differentiable
2. $I(\varphi) \approx [\hat{T}'(-\infty), \hat{T}'(\infty)]$
3. μ_q an ergodic equilibrium state for $q\hat{\varphi}$
4. $\hat{\alpha} = \hat{T}'(q) \Rightarrow \int \hat{\varphi} \, d\mu_q = \hat{\alpha} \Rightarrow \int \varphi \, d\mu_q = \alpha \approx \hat{\alpha} \Rightarrow \mu(K(\alpha)) = 1$
5. $\hat{\alpha}$ takes all values in $[\hat{T}'(-\infty), \hat{T}'(\infty)] \Rightarrow \alpha$ takes all values in $[\alpha_1, \alpha_2] \approx [\hat{T}'(-\infty), \hat{T}'(\infty)] \approx I(\varphi)$

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