Dimension Theory, Thermodynamics, & Multifractals

* many important dynamical invariants can be interpreted as dimensional quantities

* goal: ① formulate cohesive framework for dimensional characteristics, relate various characteristics to one another using thermodynamic formalism (Bowen's formula)

② outline multifractal analysis

③ relate multifractals to thermodynamics


IFS

repeller

I Dimension Theory

Motivation: length, area don't work for everything; $\#S(Z,\varepsilon)\, \varepsilon^d \to$ measure

A dimensions of sets

1. Box dimension. Defn| Given $Z \subset \mathbb{R}^d$ (or any sep. met sp) and $\varepsilon > 0$, let $S(Z,\varepsilon) \subset Z$ be a minimal ε-spanning set – that is, $Z \subset \bigcup_{x \in S(Z,\varepsilon)} B(x,\varepsilon)$, and no proper subset of $S(Z,\varepsilon)$ has this property. The box dimension of Z is

$$\dim_B Z = \lim_{\varepsilon \to 0} \frac{\log \# S(Z,\varepsilon)}{-\log \varepsilon}. \qquad \boxed{(1)}$$

Eg| $Z = [0,1]^d \Rightarrow \# S(Z,\varepsilon) \approx (1/\varepsilon)^d \Rightarrow \dim_B Z = d.$

Defn| If the limit in (1) does not exist, the lower & upper limits are the lower and upper box dimensions.

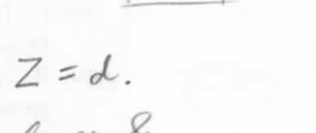
Fact| $\dim_B \overline{Z} = \dim_B Z$, so box dimension is a topologically insensitive dud. Eg| Level set of Birk av.

2. Hausdorff dimension. Allow balls of varying radii.

Defn| Given $Z \subset \mathbb{R}^d$ and $\varepsilon > 0$, consider $\boxed{(2)}$

$$\mathcal{D}(Z,\varepsilon) = \Big\{ \{(x_i, r_i)\}_{i \in I} \subset Z \times (0,\varepsilon) \;\Big|\; Z \subset \bigcup_{i \in I} B(x_i, r_i),\ I \text{ ctbl} \Big\}$$

For $\alpha \geq 0$, set $m_H(Z,\alpha,\varepsilon) = \inf_{\mathcal{D}(Z,\varepsilon)} \sum_{i \in I} r_i^{\alpha}$ $\boxed{(3)}$



and $m_H(Z,\alpha) = \lim_{\varepsilon \to 0} m_H(Z,\alpha,\varepsilon)$.

Then $\dim_H Z = \inf\{\alpha \mid m_H(Z,\alpha) = 0\} = \sup\{\alpha \mid m_H(Z,\alpha) = \infty\}$ $\boxed{(4)}$

Fact| $\dim_H Z \leq \underline{\dim}_B Z \leq \overline{\dim}_B Z \quad \forall\, Z \subset \mathbb{R}^d.$

Eg| $Z =$ middle-third Cantor set $\Rightarrow \dim_H Z = \dfrac{\log 2}{\log 3}$ (self-similarity)

3. <u>Topological entropy</u>. <u>Def'n</u>| Given X cpt met sp and $f: X \circlearrowleft$ cts, the <u>Bowen ball</u> centred at $x \in X$ of order $N \in \mathbb{N}$ and radius $\delta > 0$ is

refine dynamically

$$B(x, N, \delta) = \{ y \in X \mid d(f^k x, f^k y) \ \forall\ 0 \le k \le n \}$$

Given a cpt inv. set $Z \subset X$, and $N \in \mathbb{N}$, $\delta > 0$, let $E(Z, N, \delta) \subset Z$ be a minimal <u>(N, δ)-spanning set</u> – that is, $Z \subset \bigcup_{x \in E(Z,N,\delta)} B(x, N, \delta)$. The <u>topological entropy</u> of f on Z is

$$h_{top}(Z, f) = \lim_{\delta \to 0} \lim_{N \to \infty} \frac{\log \# E(Z, N, \delta)}{N} \qquad \boxed{(1')}$$

<u>Eg</u>| $X = S^1 \subset \mathbb{C}$, $f(z) = z^2 \Rightarrow B(z, N, \delta) = B(z, \delta 2^{-N})$

$$\Rightarrow \# E(X, N, \delta) \approx \left(\tfrac{2\pi}{\delta}\right) 2^N \Rightarrow h_{top}(X, f) = \log 2$$

do this → Similarly for interval maps. (cf. middle-third Cantor set)

<u>Remark</u>| * Can also use (N, δ)-separated sets (also true for $\dim_B$)

* Limit always exists if Z is f-inv.

What if Z is not cpt or f-inv? Then mimic (2)–(4) just as (1′) mimics (1), replacing $B(x, \varepsilon) \mapsto B(x, n, \delta)$, $\varepsilon \mapsto e^{-n}$.

$\boxed{(2')}$

<u>Defn</u>| $\mathcal{P}(Z, N, \delta) = \{ \{(x_i, n_i)\}_{i \in I} \subset Z \times [N, N+1, \dots) \mid Z \subset \bigcup_{i \in I} B(x_i, n_i, \delta) \}$

$$m_h(Z, \alpha, N, \delta) = \inf_{\mathcal{P}(Z,N,\delta)} \sum_{i \in I} (e^{-n_i})^\alpha \qquad \boxed{(3')}$$

$$m_h(Z, \alpha, \delta) = \lim_{N \to \infty} m_h(Z, \alpha, N, \delta)$$

$$h_{top}(Z, \delta) = \inf \{ \alpha \mid m_h(Z, \alpha, \delta) = 0 \} = \sup \{ \alpha \mid m_h(Z, \alpha, \delta) = \infty \}$$

$$h_{top} Z = \lim_{\delta \to 0} h_{top}(Z, \delta) \qquad \boxed{(4')}$$

<u>Fact</u>| Both definitions agree if Z is cpt & inv.

<u>Remark</u>| This illustrates general procedure of constructing <u>Carathéodory dimension characteristics</u>. (See Pesin, "Dimension Theory in Dynamical Systems" for details.)

<u>Eg</u>| Level sets of Birkhoff averages are not compact. ($\sigma: \Sigma_2^+ \circlearrowleft$)

4. Topological pressure. Def'n| Given $f : Z \circlearrowleft$ and $\varphi : Z \to \mathbb{R}$ cts, the topological pressure of φ on Z wrt. f is

$$P_Z(\varphi) = \lim_{\delta \to 0} \lim_{N \to \infty} \frac{1}{N} \log\Big(\sum_{x \in E(Z,N,\delta)} e^{S_N \varphi(x)} \Big) \qquad \boxed{(1'')}$$

where $S_N \varphi(x) = \varphi(x) + \varphi(f(x)) + \cdots + \varphi(f^{N-1}(x))$.

* Note similarity to $(1')$ - special case when $\varphi \equiv 0$.

When Z is not cpt & inv., mimic $(3')$ & $(4')$:

$$m_p(Z, \varphi, \alpha, N, \delta) = \inf_{P(Z,N,\delta)} \sum_{i \in I} e^{-n_i \alpha + S_{n_i}\varphi(x_i)}$$

$$m_p(Z, \varphi, \alpha, \delta) = \lim_{N \to \infty} m_p(Z, \varphi, \alpha, N, \delta) \qquad \boxed{(3'')}$$

$$P_Z(\varphi, \delta) = \inf\{\alpha \mid m_p(Z, \varphi, \alpha, \delta) = 0\} = \cdots \qquad \boxed{(4'')}$$

$$P_Z(\varphi) = \lim_{\delta \to 0} P_Z(\varphi, \delta)$$

5. Lyapunov exponents. Def'n| $f : X \circlearrowleft$ is conformal if the limit $a(x) = \lim_{y \to x} \frac{d(f(x), f(y))}{d(x,y)}$ exists $\forall\, x \in X$. $(a \leftrightarrow f')$

Eg| Any C^1 interval map, any rational map of $\hat{\mathbb{C}}$.

Def'n| Given f conformal and $x \in X$, the Lyapunov exponent of f at x is $\lambda(x) = \lim_{n \to \infty} \frac{1}{n} S_n(\log a)(x)$, if the limit exists.

Remark| We expect $\lambda(x)$ to relate static & dynamic quantities:

$$(*) \quad B(x, n, \delta) \approx B(x, \delta e^{-S_n(\log a)(x)}) \longrightarrow B(x, \delta e^{-n\lambda(x)})$$

Eg| f ~~tripling~~ map on $[0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$,

Λ = repeller (Cantor set). $\log a \equiv \log 3$

$\Rightarrow \lambda(x) = \log 3$ everywhere, and relates

$h_{top} \Lambda = \log 2$, $\dim_H \Lambda = \log 2 / \log 3$: $\dim = \text{ent}/\text{Lyap}$ $(**)$

6. Bowen's equation We may rewrite the relation in the last example as $h_{top} \Lambda - \lambda \cdot \dim_H \Lambda = 0$.

Fact| Variational principle: For Z cpt & inv., $\varphi : X \to \mathbb{R}$ cts,

$$P_Z(\varphi) = \sup\{ h(\mu) - \int \varphi \, d\mu \mid \mu \text{ an invariant prob. meas.} \}$$

where $h(\mu)$ is K-S entropy. (More on this later.)

II, A, 5

In example, take $\varphi = \log a \equiv \lambda$, so $\int \varphi \, d\mu = \lambda \ \forall \mu$. Thus $P_\Lambda(\log a) = [\sup_\mu h(\mu)] + \lambda = h_{top} \Lambda + \lambda$, and we have $h_{top} \Lambda - t\lambda = P_\Lambda(-t \log a)$, so $\dim_H \Lambda$ is the root of <u>Bowen's equation</u> $P_\Lambda(-t \log a) = 0$.

<u>History</u> 1979 (Bowen). "Hausdorff dimension of quasicircles" — <u>certain cpt inv. sets for fractional linear transformations of Riemann sphere</u>

1982 (Ruelle). Z maximal cpt inv., f $C^{1+\alpha}$ conformal, <u>topologically</u> mixing, uniformly expanding

1997 (~~Gatzouras~~, Peres). f can be C^1.

2000 (Barreira, Schmeling). Z <u>arbitrary</u>, f C^1 conformal, uniformly expanding

2009 (C.). Positive Lyap exp suffices, provided no <u>critical points</u>. Key <u>tool</u> is precise version of $(*)$ — can be <u>extended</u> to maps $\bar{c}$ critical pts under certain conditions.

~~<u>Remark</u> If Birkhoff averages on Z lie in $[\alpha, \beta]$, then graph of $t \mapsto P(t\varphi)$ lies in cones between lines with slopes α, β.~~

$\boxed{B}$ dimensions of measures

1. <u>local quantities</u>. <u>Defn</u> Let μ be a prob. meas. on a metric space X. The <u>pointwise dimension</u> of μ at x is

$$d_\mu(x) = \lim_{\varepsilon \to 0} \frac{\log \mu(B(x,\varepsilon))}{\log \varepsilon} \quad \left(\text{so } \mu(B(x,\varepsilon)) \approx \varepsilon^{d_\mu(x)}\right)$$

(or $\underline{d}_\mu(x)$ & $\overline{d}_\mu(x)$ if the limit does not exist).

Eg Lebesgue

Eg Σ_2^+, Bernoulli

Let μ be inv. for $f: X \circlearrowleft$. The <u>local entropy</u> of μ is

$$h_\mu(x) = \lim_{\delta \to 0} \lim_{n \to \infty} -\frac{1}{n} \log \mu(B(x,n,\delta))$$

2. <u>relationship to dimensions of sets</u>. <u>Fact 1</u> If $\mu(Z) > 0$ and $\underline{d}_\mu(x) \geq \alpha \ \forall x \in Z$, then $\dim_H(Z) \geq \alpha$.

<u>Fact 2</u> If $\mu(Z) > 0$ and $\underline{d}_\mu(x) \leq \alpha \ \forall x \in Z$, then $\dim_H Z \leq \alpha$.

(requires Besicovitch Covering Lemma)

Similar results hold for h_μ, so local quantities can be used to find the Hausdorff dimension and ~~topological~~ entropy of sets. Provided, that is, that we can find the right measure – which is the job of thermodynamic formalism.

3. <u>global quantities</u>. <u>Def'n</u> μ is <u>exact-dimensional</u> if $d_\mu(x)$ exists and is constant μ-a.e. In this case we write the common value as $\dim_H \mu$, and observe that $\dim_H \mu = \inf \{ \dim_H Z \mid \mu(Z) = 1 \}$ (which is the usual definition).

   <u>Fact</u> Hyperbolic measures are exact-dim (Barreira, Pesin, Schmeling, 1999)

   <u>Fact</u> Ergodic measures have essentially constant local entropies (Brin, Katok, 1983), and $h_\mu(x) = h(\mu)$ μ-a.e.

   Thus as above, $h(\mu) = \inf \{ h_{top} Z \mid \mu(Z) = 1 \}$

   This gives a dimensional interpretation of the KS entropy (for <u>ergodic</u> measures)

   <u>Def'n</u> Lyap exp of measure is $\lambda(\mu) = \int \log a \, d\mu$ (cf. BET)

4. <u>relationships</u>. Using (*), we have the heuristic relation

   $$d_\mu(x) = h_\mu(x) / \lambda(x) \Rightarrow \dim_H \mu = \frac{h(\mu)}{\lambda(\mu)} \quad (\text{cf. } (**))$$

   These give bounds on dimension of sets:

   $$\dim_H Z \geq h(\mu) / \lambda(\mu) \quad \forall \text{ erg. inv. } \mu$$

   Equality $\Rightarrow$ measure of maximal dimension

   But how to find this measure?

$\boxed{\text{II}}$ Thermodynamics & multifractals

$\boxed{\text{A}}$ The pressure function.

1. <u>Thermodynamic formalism</u> studies the properties of the pressure function $\varphi \mapsto P_Z(\varphi)$, $C(X) \to \mathbb{R}$.
   In particular, 1-param families of potentials: $t \mapsto P_Z(\psi + t\varphi)$

2. Variational principle. X cpt, $f : X \circlearrowleft$, $\varphi \in C(X)$,

$\mathcal{M}_f(Z) = \{\mu \mid \mu$ is an f-inv. prob. meas. on $Z\}$

$\mathcal{M}_f^E(Z) = \{\mu \in \mathcal{M}_f(Z) \mid \mu$ ergodic$\}$

Fact| $P_X(\varphi) = \sup\{h(\mu) + \int \varphi \, d\mu \mid \mu \in \mathcal{M}_f(X)\}$

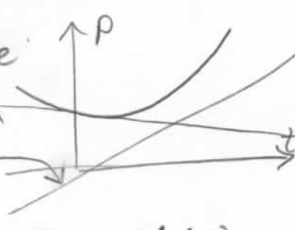
(or over $\mathcal{M}_f^E(X)$)

WARNING May not be true if we replace X by $Z \subset X$ not cpt & inv.

Def'n| A measure achieving the supremum is an equilibrium state for φ.

3. Heuristic correspondence: $\varphi \longmapsto P_Z(\varphi)$ is convex

hence graph of $t \mapsto P_Z(\psi + t\varphi)$ looks like



For convex functions, subdifferentials are important:

linear functionals $\rightarrow$ in this case, measures

Observe: $P_Z(t\varphi) \geq h(\mu) + t \int \varphi \, d\mu \quad \forall \mu \in \mathcal{M}_f(X)$

The RHS is linear in t for each fixed μ (see picture)

If equality, then we have an equilibrium state.

So (sub)differentials of $P \longleftrightarrow$ equilibrium states

differentiability $\longleftrightarrow$ uniqueness

4. Generic points: Given $\mu \in \mathcal{M}_f^E(X)$, the set of generic points for μ is $G_\mu = \{x \in X \mid \frac{1}{n} S_n \varphi(x) \longrightarrow \int \varphi \, d\mu \quad \forall \varphi \in C(X)\}$

$= \{x \in X \mid \frac{1}{n} \sum_{k=0}^{n-1} \delta_{f^k(x)} \xrightarrow{w^*} \mu\}$

Birkhoff: $\mu(G_\mu) = 1$. Bowen: $h(\mu) = h_{top}(G_\mu)$

Pesin: $P_{G_\mu}(\varphi) = h(\mu) + \int \varphi \, d\mu \Rightarrow P_{G_\mu}(t\varphi) = h(\mu) + t\int \varphi \, d\mu$

Fact| If $\frac{1}{n} S_n \varphi(x) \rightarrow \alpha \quad \forall x \in Z$, then $P_Z(\psi + t\varphi) = P_Z(\psi) + t\alpha$

In particular, $P_Z(t\varphi) = h_{top}(Z) + t\alpha$, so the graph is just a line!

5. Countable stability & an incorrect argument.

$\dim_H \left(\bigcup_{n=1}^{\infty} Z_n \right) = \sup_{n \in \mathbb{N}} \dim_H Z_n$

* holds for any dimensional characteristic
* The thing which let us find $\dim_H$ from h_{top} & λ was that λ was constant everywhere. Now λ varies so decompose:

$$X_\alpha = \{ x \in X \mid \lambda(x) = \lim_{n\to\infty} \tfrac{1}{n} S_n(\log a)(x) = \alpha \}$$

By the above fact, $P_{X_\alpha}(-t \log a) = h_{top} X_\alpha - t\alpha$

This $= 0$ when $t = h_{top}/\alpha$ = entropy / Lyap

In fact, this is $\dim_H X_\alpha$ (more on this momentarily)

Multifractal decomposition

$$X = \left(\bigcup_{\alpha \in \mathbb{R}} X_\alpha \right) \cup X' \leftarrow \text{exceptional set}$$

Idea:

$$\begin{aligned}
\dim_H X &= \sup_\alpha \dim_H X_\alpha \\
&= \sup_\alpha \{ t \mid P_{X_\alpha}(-t \log a) = 0 \} \\
&= t \text{ st. } \left(\sup_\alpha P_{X_\alpha}(-t \log a) = 0 \right) \\
&= t \text{ st. } P_X(-t \log a) = 0
\end{aligned}$$

Argument is invalid (unctbl partition + exceptional set) but result is true ... Bowen's equation

(give history here)

Eg | Compute Hausdorff dimension of repeller of

– in principle, this gives a method.

6. <u>Moran's formula</u>. Prehistory – $\rho_1, \dots, \rho_k$ ratio coefficients, construct Cantor set in $\mathbb{R}^d$.
$\dim_H C = t$ s.t. $\rho_1^t + \dots + \rho_k^t = 1$
(Use scaling: $C = \bigcup_{i=1}^k C_i$, $C_i = \rho_i C$ assuming $f_i : \mathbb{R}^d \circlearrowleft$ are contracting similarity trans.)
Now expansion is ρ_i^{-1}, so $\log a = -\log \rho_i$
$\Rightarrow \rho_1^t + \dots + \rho_k^t = \sum_{i=1}^k e^{-t \log a(x_i)} \ (= 1)$
a partition function! (n, ε)-sep set contains 1 rep from each n-cylinder, so

$$\sum_{(w_1,\dots,w_n)} e^{-t(S_n \log a)(x_{w_1 \dots w_n})} = \sum_{(w_1,\dots,w_n)} \rho_{w_1}^t \cdots \rho_{w_n}^t$$

$$= \left(\sum_i \rho_i^t\right)^n = 1 \quad \therefore \ P_C(-t \log a) = 0$$

$\boxed{B}$ Multifractal analysis

I. <u>Multifractal decomposition</u>. () is a key idea.

* local quantities in a top. dyn. sys. may vary even when constant a.e.
* decompose into level sets on which scaling or some other property is constant
* study these sets

<u>Eg</u> Birkhoff spectrum for $\sigma : \Sigma_2^+ \circlearrowleft$, $\varphi = \chi_{[0]}$

* averages may take any value from 0 to 1
* $K_\alpha^{b,\varphi} = \{x \mid \frac{1}{n} S_n \varphi(x) \longrightarrow \alpha\}$
* <u>entropy spectrum for Birkhoff averages</u>:
$$B_\varphi(\alpha) = h_{top} K_\alpha^{b,\varphi}$$
* <u>entropy spectrum for Lyap exp</u>:
$$L_E(\alpha) = h_{top} K_\alpha^{b, \log a} \leftarrow (= X_\alpha)$$

* dimension spectrum for Lyap exp

$$L_D(\alpha) = \dim_H K_\alpha^{b, \log a}$$

Eg  2 different slopes

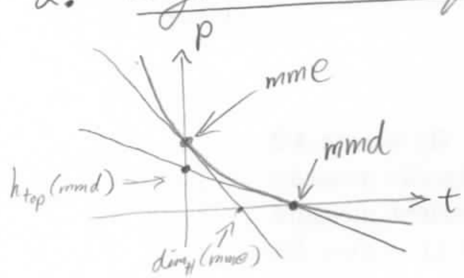
Remark | By Bowen's equation,

$$L_D(\alpha) = \frac{1}{\alpha} L_E(\alpha) \quad \text{for } \alpha > 0$$

(if no critical points)

But how to find $L_E(\alpha)$? Or $B_{\varphi}^{t}(\alpha)$?

2. Legendre transforms



$P : t \mapsto P_X(-t \log a)$ can also be described in terms of its subdifferentials. These were

$$t \longmapsto P_{X_\alpha}(-t \log a) = (h_{top} X_\alpha) - t\alpha$$

Now this line is tangent to P at the value of t for which the equilibrium state μ_t sits on X_α: $\mu_t(X_\alpha) = 1$.

At the point of tangency, we have (use $P'(t) = \int -\log a \, d\mu_t$)

$$P(t) = P_X(-t \log a) = h(\mu_t) - t\alpha = h_{top} X_\alpha - t\alpha$$

$$\therefore \quad L_E(\alpha) = h_{top} X_\alpha = P(t) + t\alpha$$

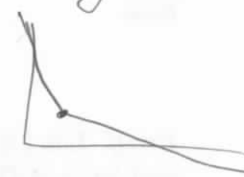
Furthermore, for any other value of t, $P(t) \geq L_E(\alpha) - t\alpha$

Thus $L_E(\alpha) = \inf_{t \in \mathbb{R}} (P(t) + t\alpha)$, the Legendre transform

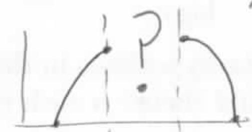
Remarkable fact – this means L_E is smooth & concave!

* Observe – global quantity (pressure) gives information about local structure

3. Non-uniform case. If f is not uniformly expanding, P may have pts of non-differentiability — phase transitions. Then it may not have ergodic equilibrium states with all slopes.



So spectrum is



General technique: L_E always $\leq$ Leg trans.
To get equality, find measures by restricting to uniformly hyperbolic subsets.

4. Critical points. $\log a$ is discontinuous!

So new techniques are needed: wk-* fails & $B(x, \delta e^{-n\lambda(x)}) \approx B(x, n, \delta)$ fail (in general). To reestablish, need slow recurrence condition: $\forall\ \varepsilon > 0, x\ \exists\ \delta$ s.t.

$$\frac{1}{n} \sum_{k=0}^{n-1} \left| \log d_\delta(f^k x, \mathrm{Crit}) \right| < \varepsilon$$

$\forall\, n$, where $d_\delta(x,y) = \begin{cases} d(x,y) & \text{if } d(x,y) < \delta \\ 0 & \text{otherwise} \end{cases}$

5. Other spectra. Can also consider entropy & dimension spectra for invariant measures. Similar results, but need Gibbs property.