Non-uniform mechanisms for stochastic behaviour in deterministic systems

Vaughn Climenhaga

University of Toronto

February 7, 2012

The talk in one slide

PHENOMENON	Deterministic systems can exhibit stochastic behaviour driven by expansion + recurrence in phase space
THEME	Statistical properties are related to uniqueness results for equilibrium states in thermodynamic formalism
KNOWN	Uniqueness holds if mechanisms driving stochasticity (expansion + recurrence) are uniform
NEW	We can adapt the results to the non-uniform setting provided obstructions to uniformity are small

Predictions in dynamical systems

Key objects:

- X = phase space for a dynamical system.
  Points in X correspond to configurations of the system.
- $f : X \circlearrowleft$ describes evolution of the state of the system over a single time step. $f^n = f \circ \cdots \circ f$ *(n times)*

Standing assumptions: X is a compact metric space, f is continuous

Predictions in dynamical systems

Key objects:

- X = phase space for a dynamical system.
  Points in X correspond to configurations of the system.
- $f \colon X \circlearrowleft$ describes evolution of the state of the system over a single time step. $f^n = f \circ \cdots \circ f$ *(n times)*

Standing assumptions: X is a compact metric space, f is continuous

Initial measurement gives neighbourhood $U \subset X$. To make a prediction based on this measurement, we must describe $f^n(U)$.

Common phenomenon: diam $f^n(U)$ becomes large relatively quickly no matter how small U is.

Predictions in dynamical systems

Key objects:

- X = phase space for a dynamical system.
  Points in X correspond to configurations of the system.
- $f: X \circlearrowleft$ describes evolution of the state of the system over a single time step. $f^n = f \circ \cdots \circ f$ *(n times)*

Standing assumptions: X is a compact metric space, f is continuous

Initial measurement gives neighbourhood $U \subset X$. To make a prediction based on this measurement, we must describe $f^n(U)$.

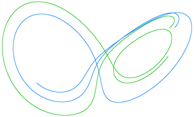
Common phenomenon: diam $f^n(U)$ becomes large relatively quickly no matter how small U is. Stronger phenomenon:

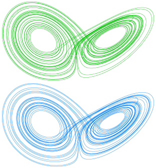
- iterates $f^n(U)$ become dense in X $\leftarrow$ *mechanism for rigorous results*

Examples

Lorenz equations (1963) – atmospheric dynamics

$$\dot{x} = \sigma(y - x) \qquad \sigma = 10$$
$$\dot{y} = x(\rho - z) - y \qquad \rho = 28$$
$$\dot{z} = xy - \beta z \qquad \beta = 8/3$$

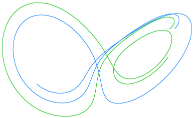


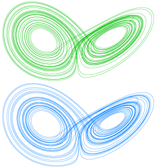


Examples

Lorenz equations (1963) – atmospheric dynamics

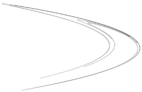
$$\dot{x} = \sigma(y - x) \qquad \sigma = 10$$
$$\dot{y} = x(\rho - z) - y \qquad \rho = 28$$
$$\dot{z} = xy - \beta z \qquad \beta = 8/3$$





Hénon map (1976) – models stretching and folding

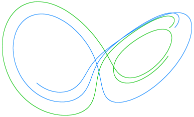
$$f(x,y) = (y + 1 - ax^2, bx) \qquad a = 1.4, b = .3$$

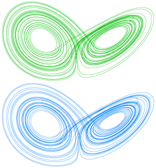


Examples

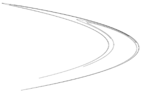
Lorenz equations (1963) – atmospheric dynamics

$$\dot{x} = \sigma(y - x) \qquad \sigma = 10$$
$$\dot{y} = x(\rho - z) - y \qquad \rho = 28$$
$$\dot{z} = xy - \beta z \qquad \beta = 8/3$$

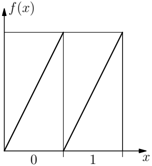




Hénon map (1976) – models stretching and folding



$$f(x, y) = (y + 1 - ax^2, bx) \qquad a = 1.4, b = .3$$

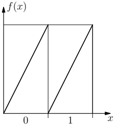


Lorenz and Hénon systems are non-uniformly hyperbolic.

Situation simplifies for (less realistic) uniformly hyperbolic systems, exemplified by the

Doubling map $f\colon S^1 \circlearrowleft,\ x \mapsto 2x \pmod 1$

Coding by symbolic systems



Doubling map $f\colon S^1 \circlearrowleft$, $x \mapsto 2x \pmod 1$

Full shift $\Sigma_2^+ = \{0,1\}^{\mathbb{N}}$, $f = \sigma\colon x_0x_1x_2\ldots \mapsto x_1x_2x_3\ldots$

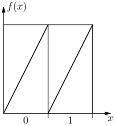
General procedure for symbolic description of dynamics:

1. Partition X as a disjoint union $A_1 \cup \cdots \cup A_d$
2. $f^n(x) \in A_{y_n}$ defines $y = \pi(x) \in \{1,\ldots,d\}^{\mathbb{N}}$
3. $\pi\colon X \to \{1,\ldots,d\}^{\mathbb{N}}$ is the coding map
4. $Y = \overline{\pi(X)}$ is the coding space

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ \pi \downarrow & & \downarrow \pi \\ Y & \xrightarrow{\sigma} & Y \end{array}$$

If $y_1 \ldots y_n = y'_1 \ldots y'_n$ *but* $y_{n+1} \neq y'_{n+1}$, *then* $d(y,y') = 2^{-n}$

Coding by symbolic systems



Doubling map $f\colon S^1 \circlearrowleft$, $x \mapsto 2x \pmod 1$

Full shift $\Sigma_2^+ = \{0,1\}^{\mathbb{N}}$, $f = \sigma\colon x_0x_1x_2\dots \mapsto x_1x_2x_3\dots$

General procedure for symbolic description of dynamics:

1. Partition X as a disjoint union $A_1 \cup \dots \cup A_d$
2. $f^n(x) \in A_{y_n}$ defines $y = \pi(x) \in \{1,\dots,d\}^{\mathbb{N}}$
3. $\pi\colon X \to \{1,\dots,d\}^{\mathbb{N}}$ is the coding map
4. $Y = \overline{\pi(X)}$ is the coding space

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ \pi\downarrow & & \downarrow\pi \\ Y & \xrightarrow{\sigma} & Y \end{array}$$

If $y_1\dots y_n = y'_1\dots y'_n$ but $y_{n+1} \neq y'_{n+1}$, then $d(y,y') = 2^{-n}$

Coding space is closed and σ-invariant: $\sigma(Y) \subset Y$.

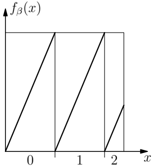
Typically many "forbidden" sequences. When is Y "good"?

β-shifts

For $\beta > 1$, Σ_β is the coding space for the map

$$f_\beta \colon [0,1] \to [0,1], \qquad x \mapsto \beta x \pmod 1$$

$1_\beta = a_1 a_2 \cdots$, where $1 = \sum_{n=1}^\infty a_n \beta^{-n}$

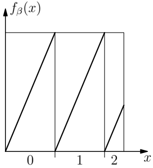


β-shifts

For $\beta > 1$, Σ_β is the coding space for the map

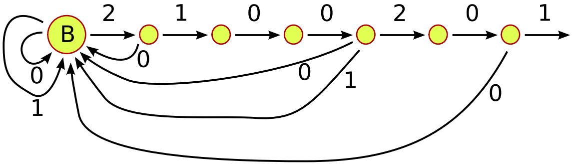
$$f_\beta \colon [0,1] \to [0,1], \qquad x \mapsto \beta x \pmod 1$$

$1_\beta = a_1 a_2 \cdots$, where $1 = \sum_{n=1}^\infty a_n \beta^{-n}$



$x \in \Sigma_\beta \quad \Leftrightarrow \quad x$ labels a walk starting at **B** $\quad \Leftrightarrow \quad \sigma^n x \preceq 1_\beta$ for all n

(Here $1_\beta = 2100201\dots$)



Invariant and ergodic measures

- $\mathcal{M} = \{\text{Borel probability measures on } X\}$

$\mu \in \mathcal{M}$ is invariant if $\int \varphi \, d\mu = \int \varphi \circ f \, d\mu$ for all $\varphi \in C(X)$

- $\mathcal{M}_f = \{\text{invariant measures}\} \subset \mathcal{M}$ *(convex, weak*-compact)*
- $\mathcal{M}_f^e = \{\text{extreme points of } \mathcal{M}_f\} = \{\text{ergodic measures}\}$

Each $\mu \in \mathcal{M}_f$ is a convex combination of ergodic measures (uniquely)

Invariant and ergodic measures

- $\mathcal{M} = \{\text{Borel probability measures on } X\}$

$\mu \in \mathcal{M}$ is invariant if $\int \varphi \, d\mu = \int \varphi \circ f \, d\mu$ for all $\varphi \in C(X)$

- $\mathcal{M}_f = \{\text{invariant measures}\} \subset \mathcal{M}$ *(convex, weak*-compact)*
- $\mathcal{M}_f^e = \{\text{extreme points of } \mathcal{M}_f\} = \{\text{ergodic measures}\}$

Each $\mu \in \mathcal{M}_f$ is a convex combination of ergodic measures (uniquely)

Theorem (G.D. Birkhoff, 1931)

If $\mu \in \mathcal{M}_f^e$ then $\frac{1}{n}\sum_{k=0}^{n-1} \varphi \circ f^k \xrightarrow{\mu\text{-a.e.}} \int \varphi \, d\mu$

The sequence $(X, \mu, \varphi \circ f^n)$ obeys the law of large numbers.

An abundance of measures

$\mathcal{M}_f^e$ is often very large. Consider $X = \Sigma_2^+$.

Periodic measures: $f^p(x) = x \rightsquigarrow \mu = \frac{1}{p}\sum_{k=1}^p \delta_{f^k(x)}$ is ergodic

- Periodic orbits are dense. *($f^p(x) = x$ has 2^p solutions)*

An abundance of measures

$\mathcal{M}_f^e$ is often very large. Consider $X = \Sigma_2^+$.

Periodic measures: $f^p(x) = x \rightsquigarrow \mu = \frac{1}{p}\sum_{k=1}^p \delta_{f^k(x)}$ is ergodic

- Periodic orbits are dense. *($f^p(x) = x$ has 2^p solutions)*

$\alpha, \beta > 0,\ \alpha + \beta = 1 \rightsquigarrow$ (α,β)-Bernoulli measure:

- $w \in \{0,1\}^n \rightsquigarrow$ cylinder set $[w] = \{x \in X \mid x_1 \cdots x_n = w\}$
- $k = \#$ of 0's in $w \Rightarrow \mu([w]) = \alpha^k \beta^{n-k}$

An abundance of measures

$\mathcal{M}_f^e$ is often very large. Consider $X = \Sigma_2^+$.

Periodic measures: $f^p(x) = x \rightsquigarrow \mu = \frac{1}{p}\sum_{k=1}^p \delta_{f^k(x)}$ is ergodic

- Periodic orbits are dense. *($f^p(x) = x$ has 2^p solutions)*

$\alpha, \beta > 0,\ \alpha + \beta = 1 \rightsquigarrow (\alpha, \beta)$-Bernoulli measure:

- $w \in \{0,1\}^n \rightsquigarrow$ cylinder set $[w] = \{x \in X \mid x_1 \cdots x_n = w\}$
- $k = \#$ of 0's in $w \Rightarrow \mu([w]) = \alpha^k \beta^{n-k}$

Also have Markov measures, Gibbs measures, etc.

How do we pick a good ergodic measure?

Entropy for shift spaces

Topological entropy of a shift space X:

- $\mathcal{L} = \{\text{words that appear in some } x \in X\}$ = language of X
- $h_{\text{top}}(X) = \lim_{n\to\infty} \frac{1}{n} \log \#\mathcal{L}_n$ $\qquad \mathcal{L}_n = \{\text{words of length } n\} \subset \mathcal{L}$

Example

$X = \Sigma_2^+ \quad \Rightarrow \quad \#\mathcal{L}_n = 2^n \quad \Rightarrow \quad h_{\text{top}}(X) = \log 2$

Entropy for shift spaces

Topological entropy of a shift space X:

- $\mathcal{L} = \{$words that appear in some $x \in X\} =$ language of X
- $h_{\text{top}}(X) = \lim_{n\to\infty} \frac{1}{n} \log \#\mathcal{L}_n$ $\qquad \mathcal{L}_n = \{$words of length $n\} \subset \mathcal{L}$

Example

$X = \Sigma_2^+ \quad \Rightarrow \quad \#\mathcal{L}_n = 2^n \quad \Rightarrow \quad h_{\text{top}}(X) = \log 2$

Measure-theoretic entropy for $\mu \in \mathcal{M}_f$:

- $h(\mu) := \lim_{n\to\infty} \frac{1}{n} \sum_{w \in \mathcal{L}_n} -\mu([w]) \log \mu([w])$

Example

Entropy of (α, β)-Bernoulli measure is $h(\mu) = -\alpha \log \alpha - \beta \log \beta$.

Variational principles

Variational principle: $h_{\mathrm{top}}(X) = \sup\{h(\mu) \mid \mu \in \mathcal{M}_f\}$

- $h(\mu) = h_{\mathrm{top}}(X) \rightsquigarrow \mu$ is a measure of maximal entropy (MME)
- Unique MME $\Rightarrow X$ is intrinsically ergodic.

Variational principles

Variational principle: $h_{\text{top}}(X) = \sup\{h(\mu) \mid \mu \in \mathcal{M}_f\}$

- $h(\mu) = h_{\text{top}}(X) \rightsquigarrow \mu$ is a measure of maximal entropy (MME)
- Unique MME $\Rightarrow X$ is intrinsically ergodic.

Generalises to topological pressure:

- Given $\varphi \in C(X)$, write $\Lambda_n(\varphi) = \sum_{w \in \mathcal{L}_n} \sup_{x \in [w]} \exp(\sum_{k=0}^{n-1} \varphi(\sigma^k x))$
- Topological pressure of φ is $P(\varphi) = \lim_{n\to\infty} \frac{1}{n} \log \Lambda_n(\varphi)$
- $P(\varphi) = \sup\{h(\mu) + \int \varphi \, d\mu \mid \mu \in \mathcal{M}_f\}$
- A measure achieving the supremum is an equilibrium state

Example: $X = \Sigma_2^+$, $\varphi(x) = s\chi_{[0]} + t\chi_{[1]}$

- $P(\varphi) = \log(e^s + e^t)$, unique eq. state is $(e^{s-P(\varphi)}, e^{t-P(\varphi)})$-Bernoulli

Unique equilibrium states

Unique equilibrium states often have strong statistical properties: central limit theorem, decay of correlations, large deviations, etc.

- the sequence of observations $(X, \mu, \varphi \circ f^n)$ has many properties in common with i.i.d. sequence of random variables

Unique equilibrium states

Unique equilibrium states often have strong statistical properties: central limit theorem, decay of correlations, large deviations, etc.

- the sequence of observations $(X, \mu, \varphi \circ f^n)$ has many properties in common with i.i.d. sequence of random variables

Joint project with D. Thompson: give efficient criteria for uniqueness of equilibrium states beyond the classical (uniform) setting

A (one-slide) digression: some applications

- **Hausdorff dimension:** If $f: M \circlearrowleft$ is conformal and J is a uniformly expanding repeller for f, then $\dim_H J = t$ solves $P_J(-t \log \|Df\|) = 0$ (R. Bowen 1979, D. Ruelle 1982). Also holds when f is non-uniformly expanding and J is any subset (C. 2011, ETDS).

A (one-slide) digression: some applications

- **Hausdorff dimension:** If $f\colon M\circlearrowleft$ is conformal and J is a uniformly expanding repeller for f, then $\dim_H J = t$ solves $P_J(-t\log\|Df\|) = 0$ (R. Bowen 1979, D. Ruelle 1982). Also holds when f is non-uniformly expanding and J is any subset (C. 2011, ETDS).
- **Multifractal analysis:** Let $K_\alpha^\varphi = \{x \mid \frac{1}{n}\sum_{k=0}^{n-1}\varphi(f^k x) \to \alpha\}$. If $T_\varphi\colon t \mapsto P(t\varphi)$ is differentiable, then the multifractal spectrum $\mathcal{B}^\varphi(\alpha) = h_{\text{top}} K_\alpha^\varphi$ is the Legendre transform of T_φ (C. 2010, thesis).

A (one-slide) digression: some applications

- **Hausdorff dimension:** If $f\colon M\circlearrowleft$ is conformal and J is a uniformly expanding repeller for f, then $\dim_H J = t$ solves $P_J(-t\log\|Df\|) = 0$ (R. Bowen 1979, D. Ruelle 1982). Also holds when f is non-uniformly expanding and J is any subset (C. 2011, ETDS).
- **Multifractal analysis:** Let $K_\alpha^\varphi = \{x \mid \frac{1}{n}\sum_{k=0}^{n-1}\varphi(f^k x) \to \alpha\}$. If $T_\varphi\colon t \mapsto P(t\varphi)$ is differentiable, then the multifractal spectrum $\mathcal{B}^\varphi(\alpha) = h_{\text{top}} K_\alpha^\varphi$ is the Legendre transform of T_φ (C. 2010, thesis).
- **Biology:** pressure can be used to distinguish between coding and non-coding DNA sequences (D. Koslicki, D. Thompson 2011)

A (one-slide) digression: some applications

- **Hausdorff dimension:** If $f\colon M\circlearrowleft$ is conformal and J is a uniformly expanding repeller for f, then $\dim_H J = t$ solves $P_J(-t\log\|Df\|) = 0$ (R. Bowen 1979, D. Ruelle 1982). Also holds when f is non-uniformly expanding and J is any subset (C. 2011, ETDS).
- **Multifractal analysis:** Let $K_\alpha^\varphi = \{x \mid \frac{1}{n}\sum_{k=0}^{n-1}\varphi(f^k x) \to \alpha\}$. If $T_\varphi\colon t \mapsto P(t\varphi)$ is differentiable, then the multifractal spectrum $\mathcal{B}^\varphi(\alpha) = h_{\mathrm{top}} K_\alpha^\varphi$ is the Legendre transform of T_φ (C. 2010, thesis).
- **Biology:** pressure can be used to distinguish between coding and non-coding DNA sequences (D. Koslicki, D. Thompson 2011)
- **SRB measures:** If f is a diffeomorphism, equilibrium states for $-\log|\det(Df|_{E^u})|$ are 'physical' measures. Uniformly hyperbolic systems: (Ya. Sinai, D. Ruelle, R. Bowen). NUH systems: (Benedicks–Carleson–Young–Wang, Alves–Bonatti–Viana, C.–Dolgopyat–Pesin).

Transitivity properties in shift spaces

For full shift $X = \Sigma_2^+$, if $v \in \mathcal{L}_n$, then $f^n([v]) = X$

- Words can be freely concatenated: $vw \in \mathcal{L}$ for all $w \in \mathcal{L}$

X is transitive if $\forall v, w \in \mathcal{L}$ there exists $u \in \mathcal{L}$ such that $vuw \in \mathcal{L}$

- Length of u may vary depending on v, w

Transitivity properties in shift spaces

For full shift $X = \Sigma_2^+$, if $v \in \mathcal{L}_n$, then $f^n([v]) = X$

- Words can be freely concatenated: $vw \in \mathcal{L}$ for all $w \in \mathcal{L}$

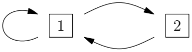
X is transitive if $\forall v, w \in \mathcal{L}$ there exists $u \in \mathcal{L}$ such that $vuw \in \mathcal{L}$

- Length of u may vary depending on v, w

Definition

X has specification if u can always be chosen to have fixed length.

$X \subset \{1, \ldots, d\}^{\mathbb{N}}$ is a subshift of finite type (SFT) if it corresponds to the set of walks on a directed graph with vertices labeled $1, \ldots, d$.



Mixing SFTs have specification

Transitivity properties in shift spaces

For full shift $X = \Sigma_2^+$, if $v \in \mathcal{L}_n$, then $f^n([v]) = X$

- Words can be freely concatenated: $vw \in \mathcal{L}$ for all $w \in \mathcal{L}$

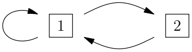
X is transitive if $\forall v, w \in \mathcal{L}$ there exists $u \in \mathcal{L}$ such that $vuw \in \mathcal{L}$

- Length of u may vary depending on v, w

Definition

X has specification if u can always be chosen to have fixed length.

$X \subset \{1, \dots, d\}^{\mathbb{N}}$ is a subshift of finite type (SFT) if it corresponds to the set of walks on a directed graph with vertices labeled $1, \dots, d$.



Mixing SFTs have specification

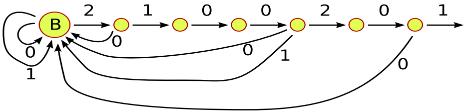
Theorem (R. Bowen, 1974)

If (X, σ) has specification then it is intrinsically ergodic.

Factors of β-shifts

Shifts without specification may have multiple MMEs

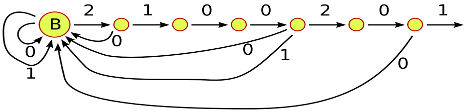
- Specification fails if 1_β contains arbitrarily long strings of 0's
- β-shifts are intrinsically ergodic (P. Walters 1978, F. Hofbauer 1979)



Factors of β-shifts

Shifts without specification may have multiple MMEs

- Specification fails if 1_β contains arbitrarily long strings of 0's
- β-shifts are intrinsically ergodic (P. Walters 1978, F. Hofbauer 1979)



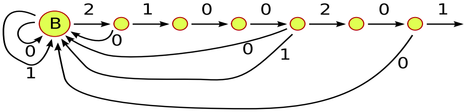
$$\begin{array}{ccc} Y & \xrightarrow{\sigma} & Y \\ \pi \downarrow & & \downarrow \pi \\ X & \xrightarrow{\sigma} & X \end{array}$$

Intrinsic ergodicity does not always pass to factors

Factors of β-shifts

Shifts without specification may have multiple MMEs

- Specification fails if 1_β contains arbitrarily long strings of 0's
- β-shifts are intrinsically ergodic (P. Walters 1978, F. Hofbauer 1979)



$$\begin{array}{ccc} Y & \xrightarrow{\sigma} & Y \\ \pi \downarrow & & \downarrow \pi \\ X & \xrightarrow{\sigma} & X \end{array}$$

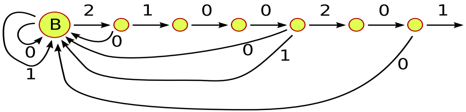
Intrinsic ergodicity does not always pass to factors

- Factors of SFTs are sofic – these are IE (B. Weiss 1970, 1973)

Factors of β-shifts

Shifts without specification may have multiple MMEs

- Specification fails if 1_β contains arbitrarily long strings of 0's
- β-shifts are intrinsically ergodic (P. Walters 1978, F. Hofbauer 1979)



$$\begin{array}{ccc} Y & \xrightarrow{\sigma} & Y \\ \pi \downarrow & & \downarrow \pi \\ X & \xrightarrow{\sigma} & X \end{array}$$

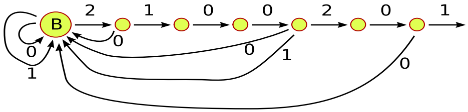
Intrinsic ergodicity does not always pass to factors

- Factors of SFTs are sofic – these are IE (B. Weiss 1970, 1973)
- Factors of shifts with specification still have specification $\Rightarrow$ IE

Factors of β-shifts

Shifts without specification may have multiple MMEs

- Specification fails if 1_β contains arbitrarily long strings of 0's
- β-shifts are intrinsically ergodic (P. Walters 1978, F. Hofbauer 1979)



$$\begin{array}{ccc} Y & \xrightarrow{\sigma} & Y \\ \pi \downarrow & & \downarrow \pi \\ X & \xrightarrow{\sigma} & X \end{array}$$

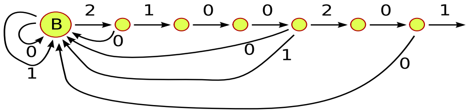
Intrinsic ergodicity does not always pass to factors

- Factors of SFTs are sofic – these are IE (B. Weiss 1970, 1973)
- Factors of shifts with specification still have specification $\Rightarrow$ IE
- Are factors of β-shifts IE? (K. Thomsen 2004, M. Boyle 2008)

Factors of β-shifts

Shifts without specification may have multiple MMEs

- Specification fails if 1_β contains arbitrarily long strings of 0's
- β-shifts are intrinsically ergodic (P. Walters 1978, F. Hofbauer 1979)



$$\begin{array}{ccc} Y & \xrightarrow{\sigma} & Y \\ \pi \downarrow & & \downarrow \pi \\ X & \xrightarrow{\sigma} & X \end{array}$$

Intrinsic ergodicity does not always pass to factors

- Factors of SFTs are sofic – these are IE (B. Weiss 1970, 1973)
- Factors of shifts with specification still have specification $\Rightarrow$ IE
- Are factors of β-shifts IE? (K. Thomsen 2004, M. Boyle 2008)

Theorem (C.–Thompson, 2010)

Yes.

Decompositions

We will be interested in decompositions of the language $\mathcal{L}$.

$\mathcal{L} = \mathcal{G}\mathcal{S} \quad \Leftrightarrow \quad$ $\mathcal{G}, \mathcal{S} \subset \mathcal{L}$ are such that every $w \in \mathcal{L}$ can be written as $w = uv$ for some $u \in \mathcal{G}$ and $v \in \mathcal{S}$

Example

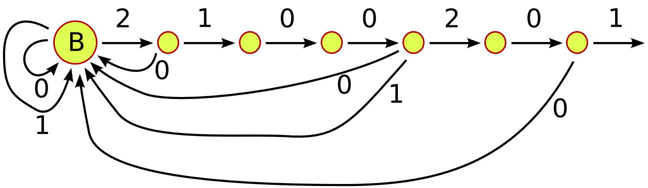
$X = \Sigma_2^+ = \{0,1\}^{\mathbb{N}}$, "good" words are $\mathcal{G} = \{\text{words ending in } 1\}$, "suffixes" are $\mathcal{S} = \{0^n \mid n \geq 0\}$

- The entropy of $\mathcal{S}$ is $h(\mathcal{S}) = \overline{\lim}_{n\to\infty} \frac{1}{n} \log \#\mathcal{S}_n$

Key observation: If $\mathcal{G}$ has specification and $h(\mathcal{S}) < h_{\mathrm{top}}(X)$, then many ideas from Bowen's proof can be adapted.

For the full shift, this is unnecessary, since $\mathcal{L}$ already has specification, but the above decomposition will return for a different reason.

Non-uniform specification for Σ_β



The only obstruction to specification is the tail of the sequence 1_β.

Let $\mathcal{G} = \{$words whose path begins and ends at **B**$\}$

- $\mathcal{G}$ has specification

Let $\mathcal{S} = \{$words whose path never returns to **B**$\}$ *(cusp excursions)*

- $\mathcal{L} = \mathcal{G}\mathcal{S}$ and $h(\mathcal{S}) = 0$

Obstructions to specification

$\mathcal{G} \subset \mathcal{L} \rightsquigarrow \mathcal{G}^M := \{vw \mid v \in \mathcal{G}, |w| \leq M\} \rightsquigarrow$ filtration $\mathcal{L} = \bigcup_M \mathcal{G}^M$

Definition

If $\mathcal{L} = \mathcal{G}\mathcal{S}$ and every $\mathcal{G}^M$ has specification, we say that $\mathcal{S}$ contains all obstructions to specification.

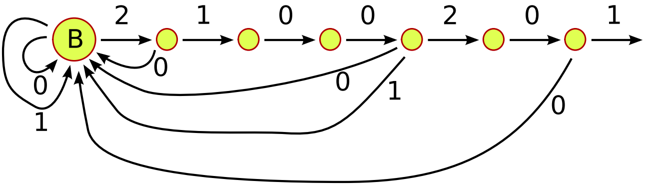
Obstructions to specification

$\mathcal{G} \subset \mathcal{L} \rightsquigarrow \mathcal{G}^M := \{vw \mid v \in \mathcal{G}, |w| \leq M\} \rightsquigarrow$ filtration $\mathcal{L} = \bigcup_M \mathcal{G}^M$

Definition

If $\mathcal{L} = \mathcal{G}\mathcal{S}$ and every $\mathcal{G}^M$ has specification, we say that $\mathcal{S}$ contains all obstructions to specification.

For the β-shift, $\mathcal{G}^M$ corresponds to walks ending on one of the first M vertices. Can return from these vertices to the base vertex in uniform time, so each $\mathcal{G}^M$ has specification.



Intrinsic ergodicity for factors

Theorem (C.–Thompson 2010)

If X is a shift space, $\mathcal{S}$ contains all obstructions to specification, and $h(\mathcal{S}) < h_{\text{top}}(X, \sigma)$, then (X, σ) is intrinsically ergodic.

Remark: If the shadowing orbits can be taken to be periodic, then the unique MME is the limit of the measures $\mu_n = \delta_{\text{Per}(n)}$.

Proposition

$\pi: X \to Y$ a factor map, $\mathcal{S} \subset \mathcal{L}(X)$ contains all obstructions $\Rightarrow$

- *$\pi(\mathcal{S}) \subset \mathcal{L}(Y)$ also contains all obstructions*
- $h(\pi(\mathcal{S})) \leq h(\mathcal{S})$

Corollary: every subshift factor of a β-shift is intrinsically ergodic

Unique equilibrium states

Theory of equilibrium states generalises theory of MMEs

- To get analogous results, control the distortion of φ:
  $V(\varphi) = \sup\{|\sum_{k=0}^{n-1} \varphi(f^k x) - \varphi(f^k y)| \mid x, y \in [w], w \in \mathcal{L}_n\}$

Theorem (Bowen 1974)

If (X, σ) has specification and $V(\varphi) < \infty$, then φ has a unique eq. state.

Unique equilibrium states

Theory of equilibrium states generalises theory of MMEs

- To get analogous results, control the distortion of φ:
  $V(\varphi) = \sup\{|\sum_{k=0}^{n-1} \varphi(f^k x) - \varphi(f^k y)| \mid x, y \in [w], w \in \mathcal{L}_n\}$

Theorem (Bowen 1974)

If (X, σ) has specification and $V(\varphi) < \infty$, then φ has a unique eq. state.

- If $\mathcal{L} = \mathcal{G}\mathcal{S}$ and $V(\varphi|_{\mathcal{G}}) < \infty$, then $\mathcal{S}$ contains all obstructions to bounded φ-distortion

Theorem (C.–Thompson 2011)

Let X be a shift space and $\varphi \in C(X)$. If $\mathcal{S}$ contains all obstructions to specification and bounded φ-distortion, and if $P(\mathcal{S}, \varphi) < P(X, \varphi)$, then there is a unique equilibrium state for φ.

Potentials on β-shifts

Bowen's result does not apply to Σ_β. Instead, the following classes of potentials have been shown to have unique equilibrium states.

- Lipschitz potentials (P. Walters 1978)
- $V(\varphi) < \infty$ and $\sup \varphi - \inf \varphi < h_{\text{top}}$ (Hofbauer–Keller 1982)

Potentials on β-shifts

Bowen's result does not apply to Σ_β. Instead, the following classes of potentials have been shown to have unique equilibrium states.

- Lipschitz potentials (P. Walters 1978)
- $V(\varphi) < \infty$ and $\sup \varphi - \inf \varphi < h_{\text{top}}$ (Hofbauer–Keller 1982)

Our results apply whenever $V(\varphi) < \infty$ and $P(\mathcal{S}, \varphi) < P(\varphi)$.

- $P(\mathcal{S}, \varphi) = \overline{\lim} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(\sigma^k 1_\beta)$
- Inequality follows from existence of exponentially many words sufficiently close to 1_β in Hamming metric

Theorem (C.–Thompson 2011)

Every Hölder continuous φ on a β-shift has a unique equilibrium state.

Potentials on β-shifts

Bowen's result does not apply to Σ_β. Instead, the following classes of potentials have been shown to have unique equilibrium states.

- Lipschitz potentials (P. Walters 1978)
- $V(\varphi) < \infty$ and $\sup \varphi - \inf \varphi < h_{\mathrm{top}}$ (Hofbauer–Keller 1982)

Our results apply whenever $V(\varphi) < \infty$ and $P(\mathcal{S}, \varphi) < P(\varphi)$.

- $P(\mathcal{S}, \varphi) = \overline{\lim} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(\sigma^k 1_\beta)$
- Inequality follows from existence of exponentially many words sufficiently close to 1_β in Hamming metric

Theorem (C.–Thompson 2011)

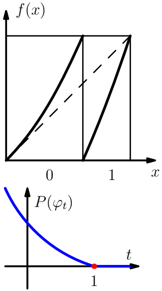
Every Hölder continuous φ on a β-shift has a unique equilibrium state.

To apply to other interval maps, need similar results when combinatorics are more complicated. Partial results available (C.–Cyr).

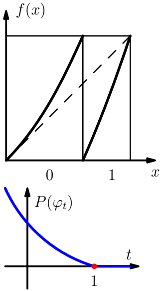
Non-uniform expansion I

Manneville–Pomeau map:

- $f(x) = x + x^{1+\varepsilon} \pmod 1$
- $\varphi_t = -t \log f'$ has unbounded distortion
- $t < 1$: unique eq. state, fully supported
- $t > 1$: unique eq. state, atomic
- $t = 1$: two equilibrium states



Non-uniform expansion I



Manneville–Pomeau map:

- $f(x) = x + x^{1+\varepsilon} \pmod 1$
- $\varphi_t = -t \log f'$ has unbounded distortion
- $t < 1$: unique eq. state, fully supported
- $t > 1$: unique eq. state, atomic
- $t = 1$: two equilibrium states

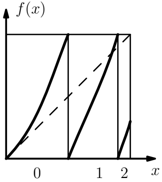
Usual approach uses inducing schemes. Our approach gives a new proof.

1. φ_t has bounded distortion on $\mathcal{G} = \{\text{words ending with } 1\}$
2. $\mathcal{S} = \{0^n \mid n \geq 0\} \Rightarrow \mathcal{L} = \mathcal{G}\mathcal{S}$
3. $P(\mathcal{S}) = \varphi(0)$, so unique equilibrium state whenever $P(\varphi) > \varphi(0)$

Non-uniform expansion II

Manneville–Pomeau β-transformation:

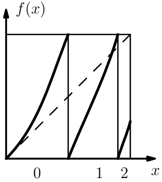
- $\gamma > 0 \rightsquigarrow f(x) = x + \gamma x^{1+\varepsilon} \pmod 1$
- $\varphi_t = -t \log f'$ has unbounded distortion
- Coding space is Σ_β for some $\beta > 1$.
- For most values of γ, specification fails.



Non-uniform expansion II

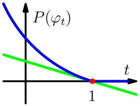
Manneville–Pomeau β-transformation:

- $\gamma > 0 \rightsquigarrow f(x) = x + \gamma x^{1+\varepsilon} \pmod 1$
- $\varphi_t = -t \log f'$ has unbounded distortion
- Coding space is Σ_β for some $\beta > 1$.
- For most values of γ, specification fails.



Theorem (C.–Thompson 2011)

For $t < 1$ and $\varepsilon \in (0,1)$, there is a unique equilibrium state for φ_t.



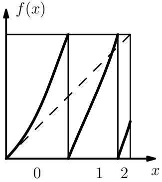
Take $\mathcal{G}, \mathcal{S}$ as for the β-shift and let

- $\hat{\mathcal{S}} = \{w0^k \mid w \in \mathcal{S} \cup \sigma\mathcal{S}, k \geq 0\}$
- $\hat{\mathcal{G}} = \{w \in \mathcal{G} \mid w_{|w|} > 0\} \cup \{w\lfloor\beta\rfloor \mid w \in \mathcal{G}\}$
- $P(\hat{\mathcal{S}}, \varphi_t) < P(\varphi_t) \Leftrightarrow P(\varphi_t) > \varphi_t(0) = 0$

Non-uniform expansion II

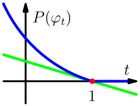
Manneville–Pomeau β-transformation:

- $\gamma > 0 \rightsquigarrow f(x) = x + \gamma x^{1+\varepsilon} \pmod 1$
- $\varphi_t = -t \log f'$ has unbounded distortion
- Coding space is Σ_β for some $\beta > 1$.
- For most values of γ, specification fails.



Theorem (C.–Thompson 2011)

For $t < 1$ and $\varepsilon \in (0,1)$, there is a unique equilibrium state for φ_t.



Take $\mathcal{G}, \mathcal{S}$ as for the β-shift and let

- $\hat{\mathcal{S}} = \{w0^k \mid w \in \mathcal{S} \cup \sigma\mathcal{S}, k \geq 0\}$
- $\hat{\mathcal{G}} = \{w \in \mathcal{G} \mid w_{|w|} > 0\} \cup \{w\lfloor\beta\rfloor \mid w \in \mathcal{G}\}$
- $P(\hat{\mathcal{S}}, \varphi_t) < P(\varphi_t) \Leftrightarrow P(\varphi_t) > \varphi_t(0) = 0$