Unique equilibrium states using regular collections of times

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The talk in one slide

Let X be a compact metric space, $f\colon X \to X$ a continuous map, and $\varphi \in C(X)$ a potential function.

Theorem (Bowen 1974)

Suppose f is expansive and has specification, and that φ has the Bowen property. Then φ has a unique equilibrium state.

Theorem (C.–Thompson 2011)

If there exists $\varepsilon > 0$ such that $P^\perp_{\mathrm{exp}}(28\varepsilon) < P(\varphi)$ and $P^\perp_{\mathrm{spec},\varphi}(\varepsilon) < P(\varphi)$, then φ has a unique equilibrium state.

Example

Certain partially hyperbolic systems.

Topological pressure

Topological dynamical system:

- X a compact metric space, $f: X \to X$ continuous, $\varphi \in C(X)$

Definition

$$\Lambda_n(\varphi, \varepsilon) = \sup \left\{ \sum_{x \in E_n} e^{S_n\varphi(x)} \,\middle|\, E_n \text{ is } (n, \varepsilon)\text{-separated} \right\}$$

Topological pressure: $P(\varphi) = \lim_\varepsilon \lim_n \frac{1}{n} \log \Lambda_n(\varphi, \varepsilon)$

- $\mathcal{M} = \{\text{Borel } f\text{-invariant probability measures on } X\}$

Variational principle: $P(\varphi) = \sup_{\mu \in \mathcal{M}} \left(h_\mu(f) + \int \varphi \, d\mu \right)$

Maximum achieved by equilibrium state.

When is there a unique equilibrium state?

Expansivity

- $B_n(x,\varepsilon) = \{y \in X \mid d(f^k x, f^k y) < \varepsilon \text{ for all } 0 \le k \le n\}$
- When f is a homeomorphism, the two-sided Bowen ball is $B_n^-(x,\varepsilon) = \{y \in X \mid d(f^k x, f^k y) < \varepsilon \text{ for all } -n \le k \le n\}$
- Expansivity: $\bigcap_{n\ge 0} B_n^-(x,\varepsilon) = \{x\}$

Definition

The expansive set at scale ε is $E(\varepsilon) := \{x \mid \bigcap_n B_n^-(x,\varepsilon) = \{x\}\}$.
The system is expansive if $E(\varepsilon) = X$ for some ε.

- Uniformly hyperbolic $\Rightarrow$ expansive
- Partially hyperbolic $\Rightarrow$???

Almost expansivity

Definition (Buzzi–Fisher)

$\mu \in \mathcal{M}$ is almost expansive if $\mu(E(\varepsilon)) = 1$ for some ε.

For our proof of uniqueness, it suffices to know that every equilibrium state is almost expansive.

Definition

The pressure of obstructions to expansivity at scale ε is

$$P^{\perp}_{\exp}(\varphi, \varepsilon) = \sup \left\{ h_\mu(f) + \int \varphi \, d\mu \;\middle|\; \mu \text{ is not almost expansive} \right\}.$$

We will consider systems where $P^{\perp}_{\exp}(\varphi, \varepsilon) < P(\varphi)$ for some ε.

Specification

Topological transitivity $\Rightarrow$ for every $(x_1, n_1), \dots, (x_k, n_k) \in X \times \mathbb{N}$ there exist $t_i \in \mathbb{N}$ and $x \in X$ such that for each $1 \le j \le k$,

$$f^{\sum_{i=0}^{j-1} n_i + t_i}(x) \in B_{n_j}(x_j, \varepsilon).$$

Definition

X has specification if for every $\varepsilon > 0$ there exists $\tau \in \mathbb{N}$ such that the above holds with $t_i \le \tau$.

Unif. hyp. $\Rightarrow$ specification. Part. hyp. $\Rightarrow$???

Key idea: if obstructions to specification have small pressure, they are invisible to equilibrium states

Bowen property

φ has the Bowen property at scale ε if there exists $V \in \mathbb{R}$ such that $|S_n\varphi(y) - S_n\varphi(x)| \leq V$ for all $(x, n) \in X \times \mathbb{N}$ and $y \in B_n(x, \varepsilon)$.

Specification + Bowen property $\Rightarrow$ there exists τ, V such that $\forall (x_1, n_1), \ldots, (x_k, n_k) \in X \times \mathbb{N}\ \exists t_i \leq \tau$ and $x \in X$ such that

- $f^{\sum_{i=0}^{j-1}(n_i+t_i)}(x) \in B_{n_j}(x_j, \varepsilon)$ for each $1 \leq j \leq k$
- $|S_{\sum(n_i+t_i)}\varphi(x) - \sum_i S_{n_i}\varphi(x_i)| \leq kV$

We will call this property specification with bounded φ-distortion.

Example

If f is uniformly hyperbolic and φ is Hölder, then f has specification with bounded φ-distortion.

Non-uniform specification

Space of orbit segments represented by $X \times \mathbb{N}$:

$$(x, n) \leftrightarrow (x, f(x), \dots, f^n(x)).$$

$\mathcal{G} \subset X \times \mathbb{N}$ represents a collection of times for each $x \in X$.

Definition

$\mathcal{G}$ has specification at scale ε if there exists $\tau \in \mathbb{N}$ such that for every $(x_1, n_1), \dots, (x_k, n_k) \in \mathcal{G}$ there exist $t_i \leq \tau$ and $x \in X$ such that $f^{\sum_{i=0}^{j-1} n_i + t_i}(x) \in B_{n_j}(x_j, \varepsilon)$ for each $1 \leq j \leq k$.

$\mathcal{G} \rightsquigarrow \mathcal{G}^M := \{(x, n) \mid (f^j(x), k) \in \mathcal{G}, 0 \leq j, k \leq M\}$

$\rightsquigarrow$ filtration $X \times \mathbb{N} = \bigcup_M \mathcal{G}^M$

We want each $\mathcal{G}^M$ to have specification with bounded φ-distortion.

Obstructions to specification

Definition

$(\mathcal{P}, \mathcal{G}, \mathcal{S}) \subset (X \times \mathbb{N})^3$ is a decomposition for (X, f) if $\forall (x, n) \in X \times \mathbb{N}\ \exists p, g, s \in \mathbb{N}$ such that $p + g + s = n$ and

$$(x, p) \in \mathcal{P} \qquad (f^p x, g) \in \mathcal{G} \qquad (f^{p+g} x, s) \in \mathcal{S}$$

- $P(\varphi, \varepsilon)$ is the growth rate of a sum over E_n.
- Given $\mathcal{D} \subset X \times \mathbb{N}$, consider sets E_n such that $E_n \times \{n\} \subset \mathcal{D}$. This gives $\Lambda_n(\mathcal{D}, \varphi, \varepsilon)$ and $P(\mathcal{D}, \varphi, \varepsilon)$.

Definition

The φ-pressure of obstructions to specification with bounded φ-distortion at scale ε is

$$P^{\perp}_{\mathrm{spec},\varphi}(\varphi, \varepsilon) = \inf\{P(\mathcal{P} \cup \mathcal{S}, \varphi, 3\varepsilon) \mid \exists \text{ decomposition } (\mathcal{P}, \mathcal{G}, \mathcal{S}) \text{ s.t. every } \mathcal{G}^M \text{ has specification at scale } \varepsilon\}$$

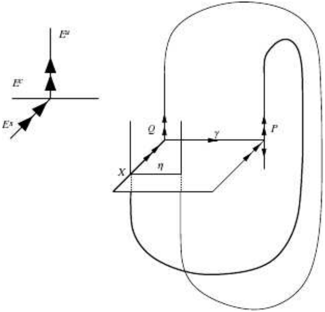
Uniqueness in the presence of small obstructions

Theorem

Let X be a compact metric space, $f : X \to X$ a continuous map, and $\varphi \in C(X)$ a potential function. If there exists $\varepsilon > 0$ such that $P^{\perp}_{\mathrm{exp}}(28\varepsilon) < P(\varphi)$ and $P^{\perp}_{\mathrm{spec},\varphi}(\varepsilon) < P(\varphi)$, then φ has a unique equilibrium state.

Partially hyperbolic horseshoes

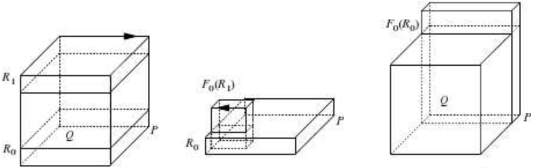
Consider a class of partially hyperbolic horseshoes in $\mathbb{R}^3$ introduced by Díaz, Horita, Rios, and Sambarino (2009).



$F_0 : \mathbb{R}^3 \to \mathbb{R}^3$ a diffeomorphism with a heterodimensional cycle: two saddles P and Q with different indices

- $W^u(P) \cap W^s(Q) \ni X$
- $W^s(P) \cap W^u(Q) \supset \gamma$

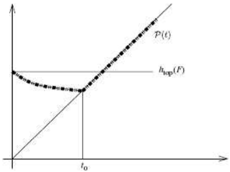
Λ = maximal invariant subset of cube



Equilibrium states

Thermodynamics studied by Leplaideur, Oliveira, and Rios (2011).

- Map is not expansive.
- Every recurrent point besides Q has negative central Lyapunov exponent.
- Equilibrium states **exist** for all $\varphi \in C(X)$.
- Pressure function for $t \mapsto P(t\varphi_c)$ is as shown, where $\varphi_c = \log |DF|_{E^c}|$.



What about uniqueness? For all small ε,

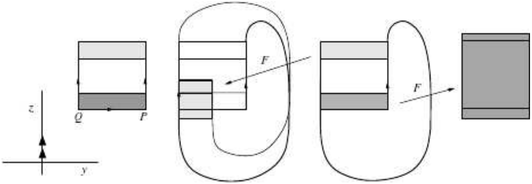
- $P^{\perp}_{\text{exp}}(\varepsilon) = -\infty$ (every invariant measure is almost expansive),
- $P^{\perp}_{\text{spec},\varphi}(\varepsilon) = \varphi(Q)$.

Theorem

$P(\varphi) > \varphi(Q)$ and φ Hölder $\Rightarrow$ φ has a unique equilibrium state

Semi-conjugacy to shift

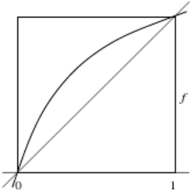
Code F by shift on $\{0,1\}$



$$\Sigma = \{\omega \in \{0,1\}^{\mathbb{Z}} \mid \omega_k \omega_{k+1} = 0 \text{ for all } k \in \mathbb{Z}\}$$

f_0 as shown, $f_1(y) = \frac{1}{3}(1-y) \Rightarrow F$ conjugate to

$$G : \Sigma \times [0,1] \to \Sigma \times [0,1]$$
$$(\omega, y) \mapsto (\sigma\omega, f_{\omega_0}(y))$$



The fixed point Q corresponds to $(\bar{0}, 0)$. The other fixed point (where E^c is contracting) corresponds to $(\bar{0}, 1)$.

Expansivity and specification

Every inv. measure has non-zero central exponent, hence almost expansive. ($P^{\perp}_{\text{exp}}(\varphi) = -\infty$)

Specification for hyperbolic systems comes from fact that $f^n W^u(x, \varepsilon)$ becomes ε-dense in the stable direction within uniform time (independent of x), and $B_s(x_2, \varepsilon) \subset B_{n_2}(x_2, \varepsilon)$.

Here central direction is stable if ω has positive frequency of 1s. So need ε-density in the W^{cs} direction.

1. Get ε-density in uniform time, but this is not quite enough!
2. Want a point $z \in B_{n_1}(x_1, \varepsilon)$ such that $f^{n_1+\tau}(z) \in B_{n_2}(x_2, \varepsilon)$; but $B_{cs}(x_2, \varepsilon) \not\subset B_{n_2}(x_2, \varepsilon)$.
3. Get $B_{cs}(x, \delta) \subset B_n(x, \varepsilon)$ as long as frequency of 1s in first n symbols of $\omega(x)$ is at least $\gamma = \gamma(\delta, \varepsilon)$.

Fix $\gamma > 0$, and put $((\omega, y), n) \in X \times \mathbb{N}$ in

- $\mathcal{P}$ if 1 occurs fewer than γn times in $\omega_1 \dots \omega_n$;
- $\mathcal{G}$ if it occurs at least γn times.

Then $(\mathcal{P}, \mathcal{G}, \emptyset)$ is a decomposition, and $P(\mathcal{P})$ is close to $\varphi(Q)$.

- **Conclusion:** $P^{\perp}_{\mathrm{spec},\varphi}(\varphi) = \inf P(\mathcal{P}) = \varphi(Q)$
- Thus if $P(\varphi) > \varphi(Q)$ for some Hölder continuous potential φ, then the main theorem applies and the equilibrium state is unique.

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Thanks for listening!

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