Tower constructions and specification properties

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Overview

Goal: Study equilibrium states in non-uniform hyperbolicity

For now: Study measure of maximal entropy in symbolic dynamics

Known:

system (X, σ)	**MME** μ
finite alphabet	existence
specification	uniqueness
mixing SFT	statistical properties

"statistical properties" $\begin{cases} (X, \psi \circ \sigma^n, \mu) \text{ has exponential decay of} \\ \text{correlations, central limit theorem, etc.} \end{cases}$

- Also comes from "tower with exponential tails": non-uniform

Questions:

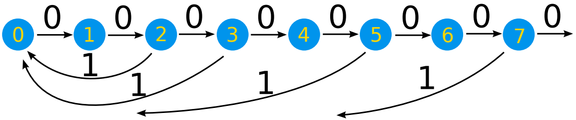
- Specification $\Rightarrow$ EDC, CLT? Non-uniform specification?

Shift spaces

Shift space: closed, shift-invariant set $X \subset A^{\mathbb{Z}}$ $\quad$ A = finite set

- Language: $\begin{cases} \mathcal{L}_n = \{w \in A^n \mid w \text{ appears in some } x \in X\} \\ \mathcal{L} = \bigcup_{n \geq 0} \mathcal{L}_n \end{cases}$
- Entropy of X is $h(\mathcal{L}) = \lim_{n\to\infty} \frac{1}{n} \log \# \mathcal{L}_n$
- Variational principle: $h(\mathcal{L}) = \sup\{h(\mu) \mid \mu \text{ shift-inv. on } X\}$

S-gap shifts: Fix $S \subset \{0, 1, 2, \dots, \}$, let $X = \overline{\{0^n 1 \mid n \in S\}^{\mathbb{Z}}}$



$\mathcal{L}$ = {finite paths on graph}

(S = {primes})

Transitivity and specification

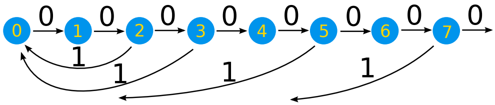
Transitive $\Leftrightarrow$ for all $u, v \in \mathcal{L}$ there exists $w \in \mathcal{L}$ s.t. $uwv \in \mathcal{L}$

- X has specification if there exists $\tau \in \mathbb{N}$ such that w can be chosen with $|w| \leq \tau$, **independently of the length of** u, v

Transitive SFTs have specification. What about non-SFTs?

S-gap shifts are transitive for every S

- Specification $\Leftrightarrow$ S has bounded gaps (syndetic)



(If $n, n+1, \ldots, n+t \notin S$ then 10^n **must** be followed by 0^t)

Uniqueness and statistical properties

μ is Gibbs if $K \leq \frac{\mu[w]}{e^{-nh(\mathcal{L})}} \leq K'$ for all $w \in \mathcal{L}_n$

Theorem (Bowen 1974)

If X has specification, then it has a unique MME μ, and μ is Gibbs.

(X, σ, μ) has exponential decay of correlations on a class of functions $\mathcal{F}$ if there is $\gamma < 1$ s.t. $\forall \varphi, \psi \in \mathcal{F}\ \exists C = C(\varphi, \psi)$ s.t.
$\left| \int (\varphi \circ \sigma^n) \psi \, d\mu - \int \varphi \, d\mu \int \psi \, d\mu \right| \leq C\gamma^n$

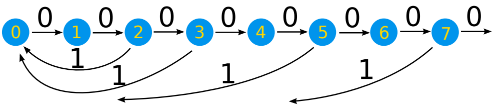
Theorem (Ruelle 1968, 1976, Sinai 1972)

If X is a mixing SFT and μ its unique MME then (X, σ, μ) has exponential decay of correlations and the CLT.

Question: Does specification imply EDC and/or CLT?

Towers with exponential tails

S-gap shifts not covered by previous slide. Can study using towers.



- $G = \{0^n 1 \mid n \in S\}$, so $G^{\mathbb{Z}} \subset X$ (indeed, $X = \overline{G^{\mathbb{Z}}}$)
- μ ergodic and $\mu \neq \delta_{\overline{0}} \Rightarrow \mu(G^{\mathbb{Z}}) = 1$

Tower is $\Omega = \{(\underline{w}, n) \in G^{\mathbb{Z}} \times \mathbb{N} \mid n \leq R(\underline{w})\}$ $\quad (R(\underline{w}) = |w_0|)$

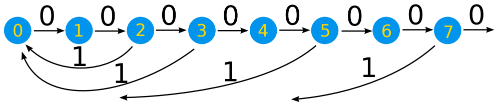
- $F : \Omega \to \Omega$ given by $F(\underline{w}, n) = \begin{cases} (\underline{w}, n+1) & n < |w_0| \\ (\sigma(\underline{w}), 1) & n = |w_0| \end{cases}$
- Get unique MME μ with exp. tails: $\mu\{R \geq n\} \leq C\gamma^n$ $(\gamma < 1)$

Young 1998: Guarantees exponential decay of correlations, CLT

Collections of words

$\mathcal{G} \subset \mathcal{L}$ has specification if there exists $\tau \in \mathbb{N}$ such that for all $u, v \in \mathcal{G}$, there exists $w \in \mathcal{L}$ with $|w| \leq \tau$ such that $uwv \in \mathcal{G}$.

- X_S: take $\mathcal{G} = \{$words starting and ending at base vertex$\}$
- More generally, if K is a finite set of vertices, take $\mathcal{G}(K) = \{$words starting and ending at vertices in $K\}$



μ has the Gibbs property on $\mathcal{G}$ if there are $K, K' > 0$ such that for all $w \in \mathcal{G}_n$ we have $K \leq \frac{\mu[w]}{e^{-nh(\mathcal{L})}} \leq K'$.

Decompositions

Idea: Unique MME if specification on "large enough" $\mathcal{G} \subset \mathcal{L}$

What does "large enough" mean?

Decomposition of $\mathcal{L}$: sets $\mathcal{C}^p, \mathcal{G}, \mathcal{C}^s \subset \mathcal{L}$ such that $\mathcal{L} = \mathcal{C}^p \mathcal{G} \mathcal{C}^s$.

$$\mathcal{G}^M = \{uvw \in \mathcal{L} \mid u \in \mathcal{C}^p, v \in \mathcal{G}, w \in \mathcal{C}^s, |u|, |w| \leq M\}$$

Write $h(\mathcal{C}^p \cup \mathcal{C}^s) := \overline{\lim}\, \frac{1}{n} \log \#(\mathcal{C}^p_n \cup \mathcal{C}^s_n)$ (entropy of obstructions)

Theorem (C.–Thompson, 2012)

Suppose $\mathcal{L}(X)$ has a decomposition such that

1. $\mathcal{G}^M$ *has specification for every* M
2. $h(\mathcal{C}^p \cup \mathcal{C}^s) < h(\mathcal{L})$

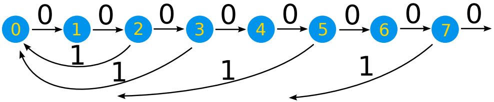
Then X has a unique MME μ. It is Gibbs on each $\mathcal{G}^M$.

Example: S-gap shifts

$\mathcal{C}^p$: paths only reaching origin at last step, or never

$\mathcal{G}$: paths starting and ending at origin

$\mathcal{C}^s$: paths starting at origin and never returning



- $\mathcal{L} = \mathcal{C}^p \mathcal{G} \mathcal{C}^s$
- $\mathcal{G}^M$: paths starting and ending in finite part of graph
- $h(\mathcal{C}^p \cup \mathcal{C}^s) = 0$

Passes to factors: every subshift factor of an S-gap shift has a unique MME. Same result for β-shifts.

Synchronised and coded shifts

Well-known: specification $\Rightarrow$ synchronised $\Rightarrow$ coded

Synchronised: $\exists\ v \in \mathcal{L}$ such that $uv \in \mathcal{L}, vw \in \mathcal{L} \Rightarrow uvw \in \mathcal{L}$

Coded: there exists $G \subset \mathcal{L}$ such that $X = \overline{G^{\mathbb{Z}}}$

- Equivalent: strongly connected countable graph presentation

Proof that synchronised $\Rightarrow$ coded: $G = \{vu \mid vuv \in \mathcal{L}\}$

- Next slides: spec $\Rightarrow$ sync ($\Rightarrow$ coded) $\Rightarrow$ tower

Dynamical interpretation: $x.v\square \leftrightarrow W^u(x.vz), \quad \square.vy \leftrightarrow W^s(\hat{z}.vy)$

- Synchronised: local product structure on $[v]$ for some v
- Markov: local product structure on $[v]$ for all (suff. long) v

A synchronising word

Specification ⇒ synchronised (Bertrand 1988). Given $u, w \in \mathcal{L}$, let

$$C(u,w) = \{y \in \mathcal{L} \mid uyw \in \mathcal{L}, |y| \leq \tau\}.$$

Specification implies non-empty.

- Start with any u, w. Note that $C(\Box u, w\Box) \subset C(u,w)$.
- Extend to $\Box u$ and $w\Box$ such that $C(\Box u, w\Box) \neq C(u,w)$.
- Iterate. $C(u,w)$ finite ⇒ process terminates.
- Let $v = uyw$ for some $y \in C(u,w) = C(\Box u, w\Box)$

Claim: v is a synchronising word

- $av \in \mathcal{L}$, $vb \in \mathcal{L} \Rightarrow auyw \in \mathcal{L}, uywb \in \mathcal{L}$
- By choice of u, w, get $y \in C(au, wb)$, so $avb = auywb \in \mathcal{L}$

Towers from specification

Specification $\Rightarrow$ unique MME μ

Also implies synchronised, hence coded with $G = \{vu \mid vuv \in \mathcal{L}\}$

- μ-a.e. x has v occur infinitely often, hence $\mu(G^{\mathbb{Z}}) = 1$
- $\{R \geq n\} \subset \{x \mid x_k \cdots x_{k+n} \not\ni v\}$ $\qquad (k = |v|)$
- this set of words grows like $e^{nh'}$ for $h' < h(\mathcal{L})$
- Gibbs property $\Rightarrow \mu\{R \geq n\} \leq Ke^{nh'}e^{-nh(\mathcal{L})} \Rightarrow$ exp. tail

Theorem (C. 2013)

If X is a shift with specification on a finite alphabet and μ is the unique MME, then μ has EDC and CLT.

Non-uniform specification

Theorem (C. 2014)

Let X be a shift with a decomposition $\mathcal{L} = \mathcal{C}^p \mathcal{G} \mathcal{C}^s$ s.t.

1. *$\mathcal{G}$ has specification;*
2. *$h(\mathcal{C}^p \cup \mathcal{C}^s) < h(\mathcal{L})$;*
3. *$uvw \in \mathcal{L}, uv \in \mathcal{G}, vw \in \mathcal{G} \Rightarrow uvw \in \mathcal{G}$.*

Then X has a unique MME μ, and (X, σ, μ) has a tower with exponential tails. In particular, μ has EDC and CLT.

Proof follows similar idea, but X need not be synchronised.

- Get a word y that synchronises $\mathcal{G}$, not $\mathcal{L}$, then build tower around 'good' returns to $[y]$, instead of all returns.

Application: every subshift factor of an S-gap shift or a β-shift has a unique MME with EDC and CLT

Non-zero potentials

Given a potential function $\varphi \colon X \to \mathbb{R}$, an *equilibrium state* is an invariant μ maximising $h(\mu) + \int \varphi \, d\mu$.

Theorem (C.–Thompson 2013)

Let X be a shift space on a finite alphabet and $\varphi \colon X \to \mathbb{R}$ be Hölder. Suppose $\mathcal{L}(X)$ has a decomposition $\mathcal{C}^p \mathcal{G} \mathcal{C}^s$ such that

1. *$\mathcal{G}^M$ has specification for every M*
2. *$P(\mathcal{C}^p \cup \mathcal{C}^s, \varphi) < P(\mathcal{L}, \varphi)$*

Then φ has a unique ES μ. It is Gibbs (for φ) on each $\mathcal{G}^M$.

Theorem (C. 2014)

If $\mathcal{G}$ has the additional property that

- *$(uvw \in \mathcal{L}, uv \in \mathcal{G}, vw \in \mathcal{G}) \Rightarrow uvw \in \mathcal{G}$*

then (X, μ) has a tower with exp. tails, so μ has EDC and the CLT.

Verifying the pressure gap

Verifying $P(\mathcal{C},\varphi) < P(\mathcal{L},\varphi)$ is more difficult when $\varphi \neq 0$

Example (Conrad 2013)

Let $X = \overline{\{0^n 1^n \mid n \in \mathbb{N}\}^{\mathbb{Z}}}$ and $\varphi = t\chi_{[1]}$. Then

- $\mathcal{L}(X)$ has a decomposition with $h(\mathcal{C}^p \cup \mathcal{C}^s) < h(\mathcal{L})$
- for large t, δ_1 is the unique ES for $t\varphi$
- there is t_0 such that $t_0\varphi$ has multiple equilibrium states

Theorem (C.–Thompson 2013)

If X is an S-gap shift or a β-shift then there is a decomposition $\mathcal{L}(X) = \mathcal{C}^p\mathcal{G}\mathcal{C}^s$ such that every Hölder continuous potential has $P(\mathcal{C}^p \cup \mathcal{C}^s,\varphi) < P(\mathcal{L},\varphi)$.

Methods are very ad hoc. Not clear how to generalise.

The non-symbolic setting

Similar results hold for non-symbolic systems: X a compact metric space, $f\colon X \to X$ continuous, $\varphi\colon X \to \mathbb{R}$ continuous.

Replace $\mathcal{L}$ with $X \times \mathbb{N}$ (space of finite orbit segments)

$$(x, n) \leftrightsquigarrow x, f(x), f^2(x), \dots, f^{n-1}(x)$$

Ask for $\mathcal{C}^p, \mathcal{G}, \mathcal{C}^s \subset X \times \mathbb{N}$ such that

- every (x, n) has $p, g, s \in \mathbb{N}_0$ such that $p + g + s = n$, $(x, p) \in \mathcal{C}^p$, $(f^p x, g) \in \mathcal{G}$, and $(f^{p+g} x, s) \in \mathcal{C}^s$
- every $\mathcal{G}^M$ has specification
- φ has the Bowen property on $\mathcal{G}$
- $P(\mathcal{C}^p \cup \mathcal{C}^s, \varphi) < P(X, \varphi)$

Together with weak expansivity condition, this gives uniqueness.

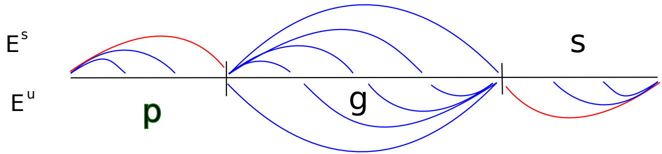
Other applications

Theorem (C.–Fisher–Thompson 2014)

For every Hölder continuous $\varphi \colon \mathbb{T}^4 \to \mathbb{R}$ there is a C^1-open set of diffeos $f \colon \mathbb{T}^4 \to \mathbb{T}^4$ (given by Bonatti and Viana) such that

- *f has a dominated splitting but is not partially hyperbolic*
- *$(\mathbb{T}^4, f, \varphi)$ has a unique equilibrium state*

$T_x\mathbb{T}^4$ splits into non-uniformly expanding and contracting E^u, E^s.



Similar approach works for geodesic flow on rank one manifolds of non-positive curvature (Burns–C.–Fisher–Thompson, in progress)