Multifractal analysis of Birkhoff averages

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Outline

1. Motivation and definitions
   - A multifractal decomposition
   - Example
2. Dimension theory
   - Quantifying level sets
   - A dynamically defined dimension
3. Results: old and new
   - Known results
   - New results
   - Other spectra

Birkhoff averages

General setting:

- X is a compact metric space;
- $f: X \to X$ is a continuous map;
- $\varphi: X \to \mathbb{R}$ is a measurable function (an observable).

Birkhoff sums: $S_n\varphi(x) = \sum_{k=0}^{n-1} \varphi(f^k(x))$.

QUESTION: What is the asymptotic behaviour of $\frac{1}{n} S_n\varphi(x)$?

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Theorem (G.D. Birkhoff)

If μ is an ergodic invariant probability measure, then $\frac{1}{n} S_n\varphi(x) \to \int \varphi \, d\mu$ for μ-a.e. $x \in X$.

This answers the question. . . but the answer depends on the measure. So what measure should we use?

Level sets

- Our setting is topological dynamics, not ergodic theory.
- There may be many invariant measures: it is not clear if one of these is preferred over others.
- So how do we understand the asymptotics of $\frac{1}{n}S_n\varphi$?

The level sets for Birkhoff averages are

$$K_\alpha = \left\{ x \in X \,\middle|\, \frac{1}{n} S_n\varphi(x) \to \alpha \right\},$$

where $\alpha \in \mathbb{R}$. (Every ergodic measure sits on a single K_α.)

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QUESTION: What are these sets like? How big are they?

So we want to examine the multifractal decomposition

$$X = \left(\bigcup_{\alpha \in \mathbb{R}} K_\alpha \right) \cup \hat{X}.$$

A symbolic example

EXAMPLE: Full shift on two symbols.

- $X = \Sigma_2^+ = \{0,1\}^{\mathbb{N}}$.
- $f = \sigma$, the shift map.
- $\varphi = \mathbf{1}_{[0]}$, the characteristic function of the 1-cylinder [0].

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OBSERVATIONS

- K_α is dense for $0 \leq \alpha \leq 1$, and empty otherwise.
- Level sets are f-invariant but not compact.

Our intuition says that K_0 is smaller than $K_{1/2}$. But in what sense?

Hausdorff dimension

The s-dimensional Hausdorff outer measure of $Z \subset X$ is

$$m_H(Z, s) = \lim_{\varepsilon \to 0} \inf_{\mathcal{D}(Z,\varepsilon)} \sum_i r_i^s,$$

where $\mathcal{D}(Z, \varepsilon) = \big\{\{(x_i, r_i)\} \mid x_i \in Z, r_i \leq \varepsilon, Z \subset \bigcup_i B(x_i, r_i)\big\}$.

The Hausdorff dimension of $Z \subset X$ is

$$\begin{aligned} \dim_H(Z) &= \sup\{s \geq 0 \mid m_H(Z, s) = +\infty\} \\ &= \inf\{s \geq 0 \mid m_H(Z, s) = 0\}. \end{aligned}$$

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If we replace "$r_i \leq \varepsilon$" with "$r_i = \varepsilon$", we get the box dimension $\dim_B Z$, which is also the growth rate of the cardinality of a minimal ε-cover.

Basic idea of multifractal analysis

GOAL: Use dimensional quantities to study the level sets K_α.

One option is the dimension spectrum for Birkhoff averages:

$$\mathcal{B}_D(\alpha) = \dim_H(K_\alpha).$$

In the example, we get $\mathcal{B}_D(0) = 0$ and $\mathcal{B}_D(1/2) = \dim_H(\Sigma_2^+)$.

- $\dim_B Z = \dim_B \overline{Z}$, and we expect level sets to be dense, so we should use Hausdorff dimension instead of box dimension.

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- $\dim_B Z = \dim_B \overline{Z}$, and we expect level sets to be dense, so we should use Hausdorff dimension instead of box dimension.
- In fact, there is another dimensional quantity that is usually easier to evaluate on the sets K_α.

Topological entropy

The Bowen ball of order n and radius δ is

$$B(x,n,\delta) = \{y \in X \mid d(f^k(x), f^k(y)) < \delta \text{ for all } 0 \leq k \leq n\}.$$

Using these, we replace $\mathcal{D}(Z,\varepsilon)$ with

$$\mathcal{P}(Z,N,\delta) = \Big\{\{(x_i,n_i)\} \mid x_i \in Z, n_i \geq N, Z \subset \bigcup B(x_i,n_i,\delta)\Big\}.$$

The s-dimensional entropy outer measure at scale δ of $Z \subset X$ is

$$m_h(Z,s,\delta) = \lim_{N\to\infty} \inf_{\mathcal{P}(Z,N,\delta)} \sum_i e^{-n_i s}.$$

The topological entropy (in the sense of Bowen) of $Z \subset X$ is

$$\begin{aligned} h_{\text{top}}(Z,\delta) &= \sup\{s \geq 0 \mid m_h(Z,s,\delta) = +\infty\} \\ &= \inf\{s \geq 0 \mid m_h(Z,s,\delta) = 0\}, \\ h_{\text{top}}(Z) &= \lim_{\delta\to 0} h_{\text{top}}(Z,\delta). \end{aligned}$$

Remarks on entropy as a dimension

We treat entropy as a dimensional quantity:

- h_{top} characterises subsets of X, not just global dynamics.

If we replace "$n_i \geq N$" with "$n_i = N$", we get the capacity entropy, which is also the growth rate (in n) of the cardinality of a minimal (n, δ)-spanning set. (Classical definition of entropy.)

- If Z is compact and invariant, the two entropies are equal.
- Capacity entropy is topologically insensitive.

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The entropy spectrum for Birkhoff averages is

$$\mathcal{B}(\alpha) = \mathcal{B}_E(\alpha) = h_{\mathrm{top}}(K_\alpha).$$

The multifractal miracle

The definition of $\mathcal{B}(\alpha)$ is roundabout, relying on

1. asymptotically defined function ($\lim \frac{1}{n} S_n\varphi$)—only measurable;
2. one-parameter family of outer measures $m_h(\cdot, s, \delta)$;
3. critical value of s for each set K_α.

Why should we get a “reasonable” function?

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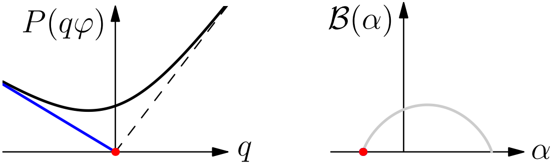
KEY IDEA: $\mathcal{B}(\alpha)$ is (often) the Legendre transform of $q \mapsto P(q\varphi)$.



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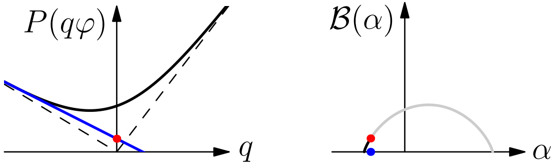

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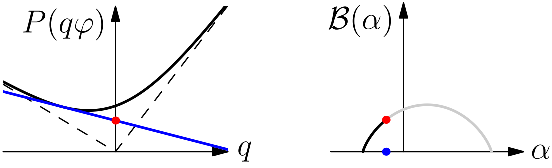

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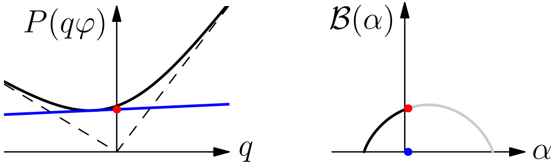

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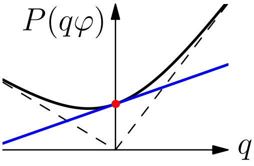

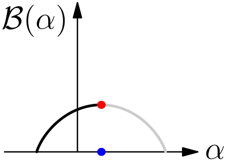

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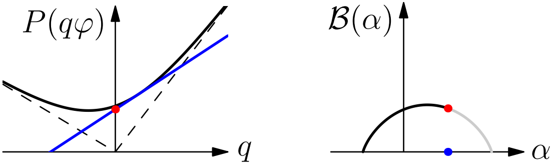

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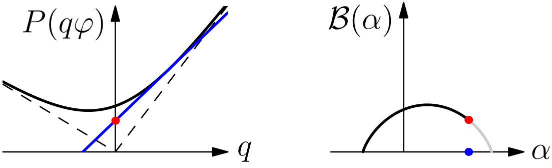

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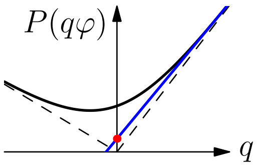

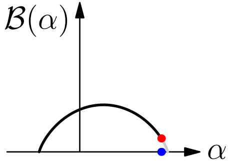

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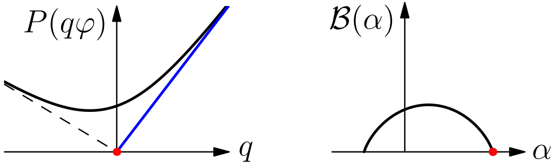

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Known results

Theorem (Pesin & H. Weiss, 2001)

Suppose the following:

- $(X, f) = (\Sigma_A^+, \sigma)$ *is a topologically mixing SFT;*
- $\varphi\colon X \to \mathbb{R}$ *is Hölder continuous;*
- φ *is not cohomologous to a constant.*

Then the multifractal formalism holds:

1. $\mathcal{B}(\alpha) = \inf_{q\in\mathbb{R}}(P(q\varphi) - q\alpha)$ *for all* $\alpha \in [\alpha_{\min}, \alpha_{\max}]$*;*
2. $K_\alpha = \emptyset$ *for all* $\alpha \notin [\alpha_{\min}, \alpha_{\max}]$*;*
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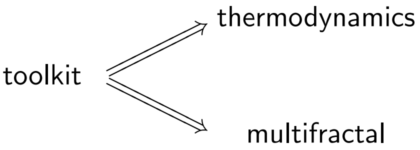
Also results by Takens & Verbitskiy, Nakaishi, Olsen, Gelfert & Rams, Iommi & Todd, Johansson, Jordan, Öberg, & Pollicott, Pfister & Sullivan on this and on $\mathcal{B}_D(\alpha)$ in more general settings.

Usual approach

1. Consider specific class of systems with a particular toolkit: Markov structure, specification, inducing scheme.
2. Use toolkit to study both thermodynamics and multifractal.



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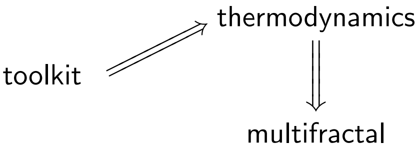
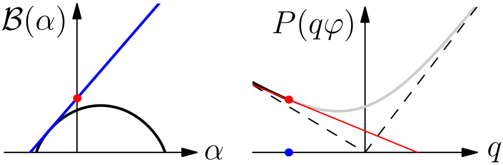


Our goal

1. Consider arbitrary continuous systems and assume thermodynamic properties are known.
2. Get multifractal as a consequence of thermodynamics.

Main result: continuous potentials

DUALITY: Pressure “should be” Legendre transform of $\mathcal{B}(\alpha)$.



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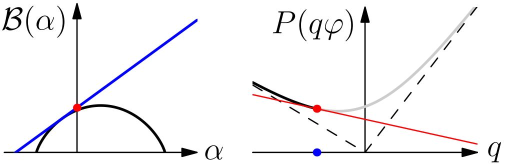

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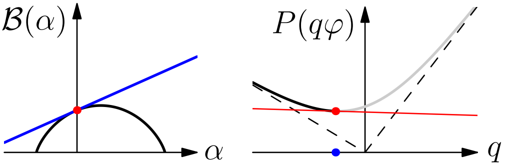

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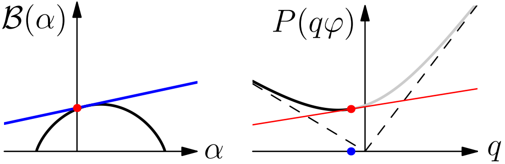

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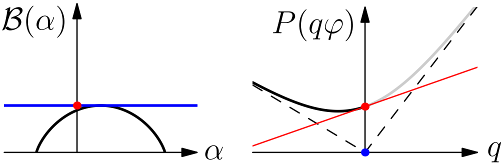

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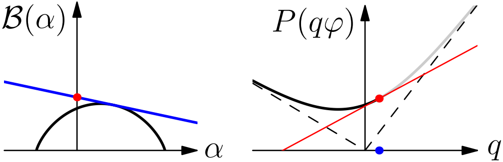

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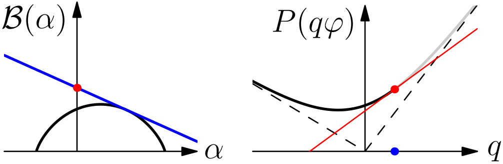

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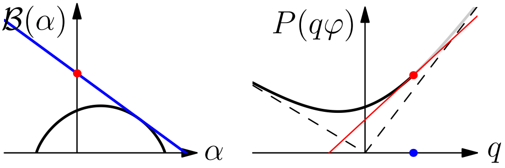

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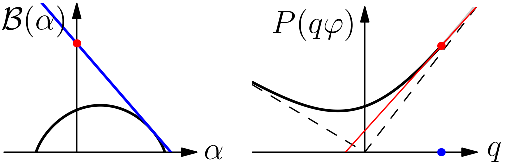

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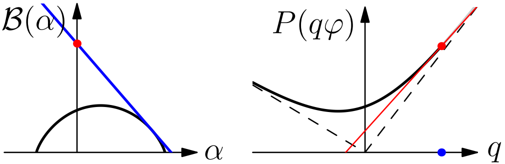

Theorem (C., 2009)

X a compact metric space, f continuous, φ continuous. Then

1. $T(q) := P(q\varphi) = \sup_{\alpha\in[\alpha_{\min},\alpha_{\max}]}(\mathcal{B}(\alpha) + q\alpha)$;
2. $K_\alpha = \emptyset$ *for all* $\alpha \notin [\alpha_{\min}, \alpha_{\max}]$.

Suppose entropy USC and T is C^1 on $Q \subset \mathbb{R}$. Then

3. $\mathcal{B}(\alpha) = \inf_{q\in\mathbb{R}}(P(q\varphi) - q\alpha)$ *for every* $\alpha \in T'(Q)$.

Sketch of proof

Main ideas

1. Mimic proof of variational principle to build measures with $h(\mu) \approx h_{\text{top}}(K_\alpha)$ and $\int \varphi \, d\mu \approx \alpha$.
2. If $x \in K_\alpha$ then empirical measures converge to a measure μ with $\int \varphi \, d\mu = \alpha$.
3. Ruelle's formula: $T'(q) = \int \varphi \, d\mu$ for an equilibrium state.

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3. Ruelle's formula: $T'(q) = \int \varphi \, d\mu$ for an equilibrium state.

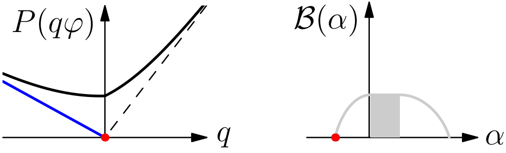
Remark: This contains a general proof that

$$\inf_{q \in \mathbb{R}} (P(q\varphi) - q\alpha) = \sup \left\{ h(\mu) \,\middle|\, \int \varphi \, d\mu = \alpha \right\},$$

the conditional variational principle used by various authors.

Phase transitions

What if T is not differentiable? Then for some α, there may not be ergodic equilibrium states with $\int \varphi \, d\mu = \alpha$.



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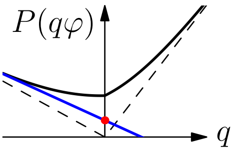

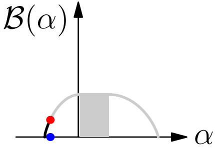

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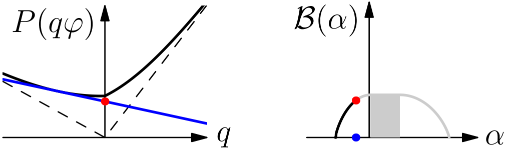

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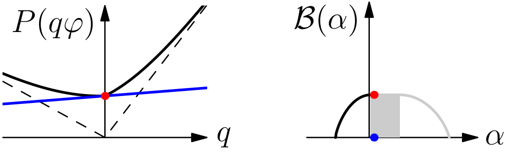

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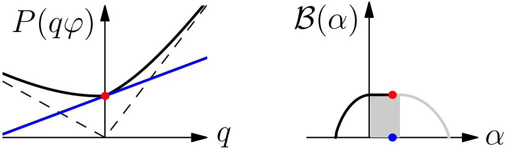

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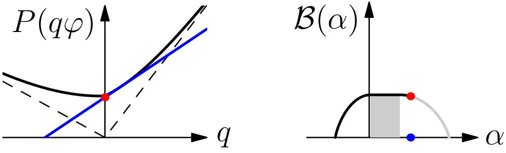

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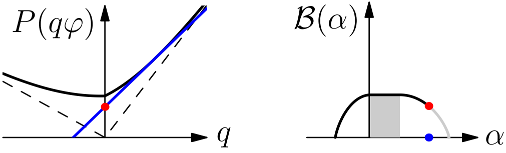

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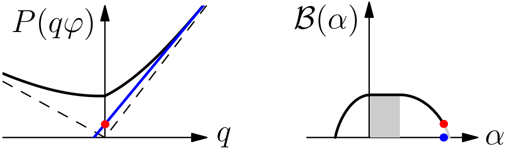

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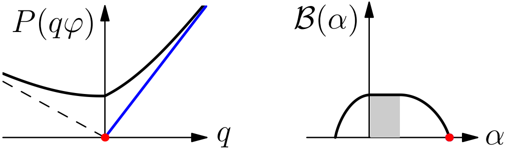

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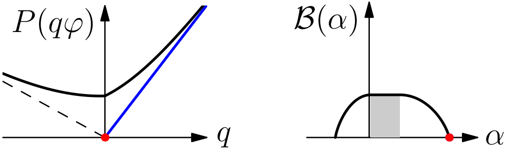

Corollary (Approximation from within)

If entropy USC and $\exists$ compact invariant $X_n \subset X$ such that

- $q \mapsto P_{X_n}(q\varphi)$ *is* C^1*;*
- $\lim_{n\to\infty} P_{X_n}(q\varphi) = P_X(q\varphi)$ *for all* $q \in \mathbb{R}$*,*

then

3. $\mathcal{B}(\alpha) = \inf_{q\in\mathbb{R}}(P(q\varphi) - q\alpha)$ *for every* $\alpha \in [\alpha_{\min}, \alpha_{\max}]$*.*

Discontinuous potentials

MORE GENERAL SITUATION: If φ is measurable and bounded, let D be the set of points of discontinuity, and let

- $h_0 = \underline{Ch}_{\text{top}}(D)$,
- $A = \{\alpha \mid \inf_q (P(q\varphi) - q\alpha) > h_0\}$,
- $Q = \{q \mid T'(q) \subset A\}$.

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Theorem (C., 2010)

X a compact metric space, f continuous, φ bounded. Then

1. $T(q) = \sup_{\alpha\in A}(\mathcal{B}(\alpha) + q\alpha)$ *for all* $q \in Q$;
2. $\mathcal{B}(\alpha) \leq h_0$ *for all* $\alpha \notin A$.

If equilibrium states exist and T is C^1 on $\tilde{Q} \subset Q$, then

3. $\mathcal{B}(\alpha) = \inf_q(T(q) - q\alpha)$ *for every* $\alpha \in T'(\tilde{Q})$.

Other multifractal spectra

1. Local asymptotic quantity $d(x)$: Birkhoff average, Lyapunov exponent, local entropy, pointwise dimension.
2. Multifractal decomposition: $K_\alpha = \{x \in X \mid d(x) = \alpha\}$.
3. Global dimensional quantity $\dim_u$: topological entropy ($u \equiv 1$), Hausdorff dimension ($u = \log |f'|$).
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LOCAL ASYMPTOTIC QUANTITY: Can be expressed as

$$d(x) = \lim_{n\to\infty} \frac{S_n\varphi(x)}{S_n\psi(x)}$$

for some functions $\varphi, \psi \colon X \to \mathbb{R}$.

GLOBAL DIMENSIONAL QUANTITY: Replace $m_h(Z, s, \delta)$ with

$$m_u(Z, s, \delta) = \lim_{N\to\infty} \inf_{\mathcal{P}(Z,N,\delta)} \sum_i e^{-(S_{n_i} u(x_i))s}.$$

General thermodynamic approach

Given $\alpha \in \mathbb{R}$, define a function $P_\alpha \colon \mathbb{R}^2 \to \mathbb{R}$ by

$$P_\alpha(q, \delta) = P(q(\varphi - \alpha\psi) - \delta u).$$

The expected multifractal spectrum associated to (φ, ψ, u) is

$$S(\alpha) = \inf\{\delta \mid \exists q \text{ s.t. } P_\alpha(q, \delta) = 0\}$$
$$= \sup\left\{ \frac{h(\mu)}{\int u \, d\mu} \,\middle|\, \frac{\int \varphi \, d\mu}{\int \psi \, d\mu} = \alpha \right\}.$$

As a Bowen's equation:

$$R_\alpha(\delta) := \inf_{q \in \mathbb{R}} P_\alpha(q, \delta), \qquad S(\alpha) = \inf\{\delta \in \mathbb{R} \mid R_\alpha(\delta) = 0\}.$$

As a generalisation of Legendre transform:

$$T_\alpha(q) := \inf\{\delta \mid P_\alpha(q, \delta) = 0\}, \qquad S(\alpha) = \inf_{q \in \mathbb{R}} T_\alpha(q).$$