Black-box Multifractal Formalism

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April 5, 2009

Outline

1. Introduction and basic concepts of the multifractal formalism

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2. Heuristics – a connection to thermodynamic formalism
   - Variational pressure
   - Legendre transforms
   - More pressure functions

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3. Known results
   - Uniformly expanding maps
   - Beyond uniformity
   - Maps with critical points
4. New results

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3. Known results
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4. New results
5. Applications and extensions
   - No critical points – continuous potentials
   - Critical points – discontinuous potentials

Local dimension

Given a compact metric space X, a continuous map $f\colon X \to X$, and an f-invariant Borel probability measure μ on X, the *local dimension* (or *pointwise dimension*) of μ at a point $x \in X$ is

$$d_\mu(x) = \lim_{r\to 0} \frac{\log \mu(B(x,r))}{\log r},$$

provided the limit exists.

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The level sets of the local dimension are

$$K_\alpha^d = \{x \in X \mid d_\mu(x) = \alpha\}.$$

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The decomposition

$$X = \left(\bigcup_{\alpha \in \mathbb{R}} K_\alpha^d \right) \cup K_*^d$$

is a *multifractal decomposition* of X.

Hyperbolic measures

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Theorem (Barreira–Pesin–Schmeling, 1999)

If μ is hyperbolic, then one of the level sets K_α^d has full μ-measure.

Thus all but one of the level sets have measure zero, and measure is not the appropriate tool to use.

Dimension spectrum

The *dimension spectrum for local dimensions* of μ is

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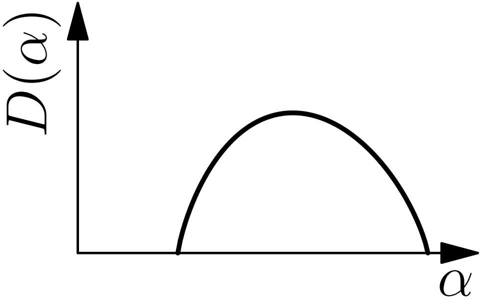
Surprising fact ("Multifractal Miracle"): In many cases of interest, $D(\alpha)$ is concave and smooth (indeed, even analytic)!



Local entropy

The *local entropy* of μ at a point $x \in X$ is

$$h_\mu(x) = \lim_{\delta \to 0} \varliminf_{n \to \infty} \frac{-\log \mu(B(x,n,\delta))}{n},$$

provided the limit exists, where

$$B(x,n,\delta) = \{y \in X \mid d(f^k(x), f^k(y)) < \delta \text{ for all } 0 \le k \le n-1\}$$

is the *Bowen ball* of radius δ and length n.

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The level sets of the local entropy are

$$K_\alpha^e = \{x \in X \mid h_\mu(x) = \alpha\},$$

and we again have an associated multifractal decomposition.

Entropy spectrum

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If μ is ergodic, then one of the level sets K^e_α has full μ-measure.

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Thus we define the *entropy spectrum for local entropies* of (μ, f) as

$$E(\alpha) = h_{\text{top}} K_\alpha^e,$$

where h_{top} is defined in the sense of Bowen, since the level sets are in general non-compact (indeed, dense).
This displays the same sort of behaviour as $D(\alpha)$.

Birkhoff spectrum

Given a function $\varphi\colon X \to \mathbb{R}$, the level sets of the Birkhoff averages are

$$K_\alpha^b = \left\{ x \in X \;\middle|\; \lim_{n\to\infty} \frac{1}{n} S_n\varphi(x) = \alpha \right\},$$

where $S_n\varphi(x) = \sum_{k=0}^{n-1} \varphi(f^k(x))$.

Theorem (G.D. Birkhoff, 1931)

If μ is ergodic, then one of the level sets K_α^b has full μ-measure.

So define the *entropy spectrum for Birkhoff averages* of (φ, f) (or just the *Birkhoff spectrum*):

$$B(\alpha) = h_{\text{top}} K_\alpha^b.$$

Lyapunov spectrum

If f is conformal (Df is a scalar multiple of an isometry), then the Lyapunov exponent of f at x is

$$\lambda(x) = \lim_{n\to\infty} \frac{1}{n} \log \prod_{k=0}^{n-1} |Df(f^k(x))| = \lim_{n\to\infty} \frac{1}{n} S_n(\log|Df|)(x),$$

provided the limit exists. Thus the Birkhoff spectrum of $\varphi = \log|Df|$ is also called the *entropy spectrum for Lyapunov exponents* of f. We may also consider the *dimension spectrum for Lyapunov exponents* of f,

$$L(\lambda) = \dim_H K_\lambda^b.$$

The variational pressure

Given a continuous potential function $\varphi\colon X \to \mathbb{R}$, the *variational pressure* of φ is

$$P(\varphi) = \sup \left\{ h_\mu(f) + \int_X \varphi \, d\mu \,\middle|\, \mu \in \mathcal{M}(X) \right\},$$

where $h_\mu(f)$ is the measure-theoretic entropy and $\mathcal{M}(X)$ is the set of f-invariant Borel probability measures on X.

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We will consider various functions defined in terms of the pressure, the simplest of which is

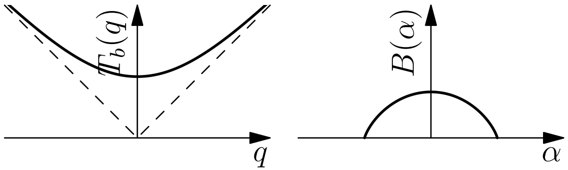
$$\begin{aligned} T_b \colon \mathbb{R} &\to \mathbb{R}, \\ q &\mapsto P(q\varphi). \end{aligned}$$

Legendre transforms

Standard approach: Show that multifractal spectra are concave and smooth by establishing a Legendre transform duality with the appropriate pressure function:

$$T_b(q) = B^{L_1}(q) = \sup_{\alpha \in \mathbb{R}} (B(\alpha) + q\alpha),$$

$$B(\alpha) = T_b^{L_2}(\alpha) = \inf_{q \in \mathbb{R}} (T_b(q) - q\alpha).$$



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$$h_\mu(x) = -\lim_{n\to\infty}\frac{1}{n}S_n\varphi_1(x).$$

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$$h_\mu(x) = -\lim_{n\to\infty}\frac{1}{n}S_n\varphi_1(x).$$

Setting $T_e(q) = T_b(-q) = P(-q\varphi_1)$, we hope to find that

$$T_e(q) = E^{L_3}(q) = \sup_{\alpha\in\mathbb{R}}(E(\alpha) - q\alpha),$$
$$E(\alpha) = T_e^{L_4}(\alpha) = \inf_{q\in\mathbb{R}}(T_e(q) + q\alpha).$$

The dimension spectrum

Conformal maps: Let $T_d(q)$ be the (hopefully unique) root of the equation

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So, in what settings can all this actually be proved?

Conformal repellers

A maximally invariant set J is a *repeller* if $|Df(v)|$ is uniformly > 1 for $|v| = 1$.

Theorem (Pesin–Weiss, 1997)

If f is $C^{1+\alpha}$ and μ is a Gibbs measure for a Hölder continuous potential φ on a conformal repeller, then the multifractal formalism holds for the dimension spectrum of μ and the Birkhoff spectrum of φ (and hence the entropy spectrum of μ as well).

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In particular, this holds for one-dimensional uniformly expanding Markov maps. Note that in one dimension conformality is automatic.

Beyond uniformity

Manneville–Pomeau maps: A class of one-dimensional Markov maps with indifferent fixed points. Pollicott and Weiss (1999), Nakaishi (2000), and Gelfert and Rams (2008) studied the multifractal formalism for the Lyapunov spectrum.

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Expansive maps with specification: Takens and Verbitski (1999) established the multifractal formalism for the entropy spectrum of Gibbs measures for sufficiently regular potentials.

Unimodal maps

The logistic family of maps is $f_a \colon [0,1] \to [0,1]$ given by

$$f_a(x) = ax(1-x).$$

For a positive Lebesgue measure set of parameters a, f_a is Collet–Eckmann (exponential growth along orbit of critical point). Thus we have some sort of hyperbolicity, but the presence of the critical point and the discontinuity of the potential $-\log|Df|$ disrupts previous approaches.

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Thermodynamic results were obtained by Bruin and Todd (2008) and Pesin and Senti (2008); multifractal results have been obtained by Todd (2008) and Todd and Iommi (2009).

Multifractal formalism as a black box

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3. Use these tools, together with the thermodynamic results, to derive multifractal results.

Multifractal formalism as a black box

What do all these have in common? The general scheme is:

1. Develop tools for specific class of maps (finite Markov partitions, inducing schemes, specification).
2. Use these tools to derive thermodynamic results.
3. Use these tools, together with the thermodynamic results, to derive multifractal results.

Idea: Suppose we know that a system has "nice" thermodynamic properties – for example, existence and uniqueness of equilibrium states for $q\varphi$ (or the appropriate potentials φ_q) over some range of q – but we do not know anything else about the system. What can we say about the multifractal formalism?

Entropy spectrum for Birkhoff averages

Consider a compact metric space X, a continuous map $f\colon X \to X$ be continuous, and a continuous function $\varphi\colon X \to \mathbb{R}$.

Theorem (C., 2009)

Let $(q_1, q_2) \subset \mathbb{R}$ be such that the following hold.

- *Existence: For every $q \in (q_1, q_2)$ there exists a (not necessarily unique) equilibrium state for the potential function $q\varphi$.*
- *Differentiability: The map $T_b\colon q \mapsto P(q\varphi)$ is $\mathcal{C}^1$ on (q_1, q_2).*

Then the Birkhoff spectrum satisfies the multifractal formalism on (α_1, α_2), where $\alpha_i = T_b'(q_i)$.

Note that we require nothing of the system beyond the thermodynamic conditions!

Gibbs measures

To deal with the entropy and dimension spectra of a measure μ in terms of the thermodynamics of φ, we need a way to link μ with φ.

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Definition

μ is a *Gibbs measure* for φ if for every $\delta > 0$ there exists $M > 0$ and $P \in \mathbb{R}$ such that

$$\frac{1}{M} \leq \frac{\mu(B(x,n,\delta))}{\exp(-nP + S_n\varphi(x))} \leq M$$

for every $x \in X$, $n \in \mathbb{N}$.

This is equivalent to demanding that

$$-\frac{1}{n}\log \mu(B(x,n,\delta)) + \frac{1}{n}S_n\varphi(x) = P + O(1/n)$$

uniformly in x.

Various sorts of Gibbs-ness

- Gibbs:
$$-\frac{1}{n}\log\mu(B(x,n,\delta))+\frac{1}{n}S_n\varphi(x)\to P$$
for every x, with uniform error term of size $O(1/n)$.
- Birkhoff and Brin–Katok: If μ is an ergodic equilibrium measure for φ, then
$$-\frac{1}{n}\log\mu(B(x,n,\delta))+\frac{1}{n}S_n\varphi(x)\to h_\mu(f)+\int\varphi\,d\mu=P(\varphi)$$
for μ*-a.e.* x, with no information on uniformity or rate of convergence.

We require convergence everywhere, but do not need any uniformity or estimates on rate of convergence.

Entropy spectrum for local entropies

Recall that $T_e(q) = P(q\varphi_1)$, where $\varphi_1 = \varphi - P(\varphi)$.

Theorem (C., 2009)

Let μ be a Gibbs measure for φ. Suppose that T_e is $\mathcal{C}^1$ on some interval (q_1, q_2) on which existence holds. Then the entropy spectrum satisfies the multifractal formalism on (α_2, α_1), where $\alpha_i = -T_e'(q_i)$.

From entropy to dimension

The analogous result for the dimension spectrum uses two important tools, both of which require f to be conformal:

- When f is conformal, the Bowen balls $B(x, n, \delta)$ are quite regular in shape and are not distorted. Heuristically, we have the following relationship between local entropy, local dimension, and Lyapunov exponent:

$$d_\mu(x) = \frac{h_\mu(x)}{\lambda(x)}.$$

- If $\log |Df|$ is continuous, then pressure and Hausdorff dimension are related by a generalisation of Bowen's equation

$$P_Z(-t \log |Df|) = 0 \iff t = \dim_H Z.$$

Dimension spectrum for Lyapunov exponents

Using the generalised Bowen's equation, we observe that when the Lyapunov exponent $\lambda(x) = \lambda$ is constant on Z, we have

$$P_Z(-t \log |Df|) = h_{\text{top}}(Z) - t\lambda = 0$$

if and only if $t = h_{\text{top}}(Z)/\lambda$. Thus the dimension spectrum for Lyapunov exponents is

$$L(\lambda) = \dim_H K_\lambda^b = \frac{1}{\lambda} h_{\text{top}}(K_\lambda^b) = \frac{1}{\lambda} B(\lambda).$$

This may not be concave!

Dimension spectrum for local dimensions

Recall that $T_d(q)$ is defined by $P(q\varphi_1 - T_d(q)\log|Df|) = 0$.

Theorem (C., 2009)

Let μ be a Gibbs measure for φ. Suppose that the following hold.

- *f is conformal.*
- *$\log|Df|$ is continuous.*
- *Every f-invariant measure has positive Lyapunov exponent.*
- *T_d is $\mathcal{C}^1$ on some interval (q_1, q_2).*
- *For every $\alpha \in \mathbb{R}$ and $q \in (q_1, q_2)$, there exists an equilibrium state for $q\varphi_1 - \alpha\log|Df|$.*

Then the dimension spectrum satisfies the multifractal formalism on (α_2, α_1), where $\alpha_i = -T_d'(q_i)$.

Birkhoff and entropy spectra

Applications using h_{top} – no dimensional requirements
For suitable continuous potentials, thermodynamic results are available for all three classes of examples (uniformly expanding, Manneville–Pomeau, Collet–Eckmann), and so the present results establish the multifractal formalism for Birkhoff averages on the corresponding part of the spectrum.

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One particularly interesting potential is $\varphi = \log |Df|$, for which the Birkhoff averages are the Lyapunov exponents, and for which the equilibrium measure is physically observable (an SRB measure).

- No critical points: φ is continuous, the present results apply.
- Critical points: φ is discontinuous. What happens?

Lyapunov and dimension spectra

For the dimensional results, we must check the non-thermodynamic requirements:

- Conformality – automatic for one-dimensional maps.
- Continuity of $\log |Df|$ – holds if f has no critical points.
- Positive Lyapunov exponents for all invariant measures – holds for Collet–Eckmann, but not Manneville–Pomeau.

Problems with discontinuities

We would like to include Lyapunov and dimension spectra for maps with critical points in this result, which requires us to deal with the case where $\varphi = \log|Df|$ is discontinuous. This creates two difficulties.

- Continuity of φ is used exactly once in each result; to guarantee that $\int \varphi \, d\mu_n \to \int \varphi \, d\mu$ when $\mu_n \stackrel{\text{wk}*}{\to} \mu$. In general, this does not hold for discontinuous φ.

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- The two results on dimensions – $d_\mu = h_\mu/\lambda$ and the generalised form of Bowen's equation – become much more subtle in the presence of critical points. They rely on the fact that the Bowen balls are somehow "well-behaved", that the diameter of $B(x, n, \delta)$ decays according to $\lambda(x)$, but the proof of this fact does not carry over to maps with critical points.

A convergence requirement

Given $x \in X$, the *empirical measures* along the orbit of x are

$$\mu_{x,n} = \frac{1}{n} \sum_{k=0}^{n-1} \delta_x \circ f^{-k}.$$

In order to establish the multifractal formalism for B for discontinuous φ using the present approach, we need the following to hold:

- If $\mu_{x,n_j} \to \mu$ and $\frac{1}{n_j} S_{n_j}\varphi(x) = \int \varphi \, d\mu_{x,n_j} \to \alpha \in \mathbb{R}$, then $\int \varphi \, d\mu = \alpha$.

The set $\mathcal{A}$

The requirements that the Bowen balls be well-behaved and that Lyapunov exponents be positive can potentially be swept under the rug by restricting our attention to a set $\mathcal{A}$ on which these both hold. Then we compute the spectrum

$$D_{\mathcal{A}}(\alpha) = \dim_H K_\alpha^d \cap \mathcal{A},$$

and hope to show that this is equal to $\dim_H K_\alpha^d$, but this has yet to be done.