Bowen's equation & multifractal analysis

$\boxed{\S 0}$ Linear Cantor sets & Moran's equation

$\boxed{\S 1}$ Non-linear Cantor sets & Bowen's equation

(A) Thermodynamic formalism & classical results

(B) Dimension theory & new results

$\boxed{\S 2}$ Multifractal analysis

(A) Birkhoff & Lyapunov spectra – defns

(B) Legendre transforms

(C) Other spectra & general picture

$\boxed{\S 0}$ Hausdorff dim: $\quad Z \subset \mathbb{R}^d, \quad \varepsilon > 0$

$$\mathcal{D}(Z,\varepsilon) = \{\{x_i, r_i\}_{i \in I} \mid x_i \in Z,\ 0 < r_i < \varepsilon,\ \bigcup_i B(x_i, r_i) \supset Z\}$$

(set of covers)

$$m_H(Z,\alpha,\varepsilon) = \inf_{\mathcal{D}(Z,\varepsilon)} \sum_{i \in I} r_i^\alpha \quad \text{(outer measure)} \quad \forall \alpha > 0$$

$$m_H(Z,\alpha) = \lim_{\varepsilon \to 0} m(Z,\alpha,\varepsilon)$$

$m(Z,\alpha)$ — (critical value) — α

$$\dim_H Z = \inf\{\alpha \mid m(Z,\alpha) = 0\} = \sup\{\alpha \mid m_H(Z,\alpha) = \infty\}$$

Box dim: $S(Z,\varepsilon)$ a <u>minimal ε-spanning set</u>,

$$\dim_B Z = \lim_{\varepsilon \to 0} \frac{\log \# S(Z,\varepsilon)}{\log(1/\varepsilon)}$$

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① Canonical example: middle-third Cantor set

$$\# S(J, 3^{-n}) = 2^n \implies \dim_B J = \frac{\log 2^n}{\log 3^n} = \frac{\log 2}{\log 3}$$

This is Hausdorff dimension as well:

σ_1, σ_2 contractions by $\frac{1}{3}$, $\quad J = \sigma_1 J \cup \sigma_2 J$

$$m_H(J,\alpha) = m_H(\sigma_1 J,\alpha) + m_H(\sigma_2 J,\alpha) \quad \text{(sort of)}$$
$$= (1/3)^\alpha m_H(J,\alpha) + (\tfrac{1}{3})^\alpha m_H(J,\alpha) = 2(\tfrac{1}{3})^\alpha m_H(J,\alpha) \Rightarrow \cdots$$

② Tweak a bit: σ_1, σ_2 contractions by $\rho_1, \rho_2 \in (0,1)$

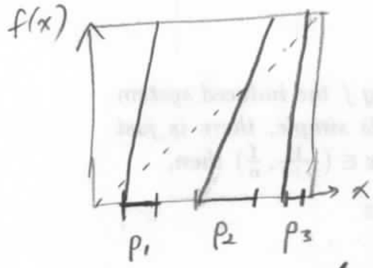
$$m_H(J,\alpha) = \rho_1^\alpha m_H(J,\alpha) + \rho_2^\alpha m_H(J,\alpha) \implies \rho_1^\alpha + \rho_2^\alpha = 1 \qquad (*)$$

In general, $\rho_1^t + \cdots + \rho_k^t = 1$ (Moran's equation) $\rightsquigarrow t = \dim_H J$

(Box dim not so obvious)

$\boxed{-2-}$

③ ~~Introduce~~ dynamics: f linear on each I_j,



$|I_j| = \rho_j, \quad D = \bigcup_j I_j$

$J = \{x \mid f^n x \in D \ \forall\ n \geq 0\}$

is a Cantor set, just as before

What if we make f non-linear?

$\boxed{\S 1}$ Write Moran's equation in a funny way:

Pick $x_j \in I_j$, then $\rho_j = 1/|f'(x_j)| = e^{-\log|f'(x_j)|}$

$\Rightarrow$ (*) becomes $\sum_j e^{-t \log|f'(x_j)|} = 1$

* Non-linear $\Rightarrow$ $f'(x_j)$ no longer gives ρ_j BUT, asymptotic behaviours are related.

$I_n^n(x) = \{y \mid f^k(x), f^k(y) \text{ in same } I_j \ \forall\ 0 \leq k \leq n\}$

$|I^n(x)| \approx |(f^n)'(x)|^{-1} = |f'(x) f'(f(x)) \cdots f'(f^{n-1}(x))|^{-1}$

$= e^{-S_n(\log|f'|)(x)}$

where $S_n \varphi(x) = \varphi(x) + \varphi(f(x)) + \cdots + \varphi(f^{n-1}(x))$

* What did $\sum_j \rho_j^t = 1$ give? Allowed us to say that

$\sum_{x \in E_n} |I^n(x)|^t = 1$, where E_n includes 1 pt from each I^n.

The sum is $\approx m_H(J, t, \varepsilon_n)$, and corresponds to

$\sum_{x \in E_n} e^{S_n \varphi(x)}$ where $\varphi = -t \log|f'|$

* $E(n, \delta)$ a minimal (n, δ)-spanning set:

$\bigcup_{x \in E} B(x, n, \delta) \supseteq J \qquad B(x, n, \delta) = $ Bowen ball

$Z(n, \delta) = \sum_E e^{S_n \varphi(x)} = $ partition fn

topological pressure $= P(\varphi) = \lim_{n \to \infty} \frac{1}{n} \log Z(n, \delta)$

* $\sum_{x \in E_n} e^{S_n(-t \log|f'|)(x)}$ remains bdd for $t = \dim_H J$

$\longrightarrow \quad P(-t \log|f'|) = 0 \qquad (**) \qquad$ (Bowen's equation)

$\boxed{\text{Thm}}$ (Ruelle, 1982)

M a Riem. mfd, $V \subset M$ open, $f: V \to M$, $C^{1+\varepsilon}$. Suppose

① f conformal: $(Df(x) = a(x)\, \mathrm{Isom}(x) \quad \forall\, x \in V)$

② J compact & invariant

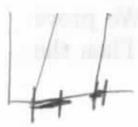
③ J maximal: $J = \{x \in V \mid f^n(x) \in V \quad \forall\, n \geq 0\}$

④ f top. mix. on J

⑤ f unif. expanding on J: $\exists\, r > 1$ st. $a(x) \geq r \quad \forall\, x$.

Then $\exists!\, t \in \mathbb{R}$ st. $P_J(-t \log a) = 0$, and for this t, we have $\dim_H J = t$. (1997 → C^1 by Gatzouras, Peres)

$\boxed{Q:}$ What if we are interested in non-compact subsets of J?

$\boxed{\text{Eg}}$  $|I_1| \neq |I_2|$ $\quad a_n(x) = \#\{0 \leq k < n \mid f^k(x) \in I_1\}$

$$Z = \{x \mid \tfrac{a_n(x)}{n} \to \tfrac{1}{2}\}$$

$(\mu(Z) = 1,\ \mu = \text{mme})$ What is $\dim_H Z$?

Want to compute as t st. $P_Z(-t \log a) = 0$...

but need proper defn of pressure.

* Usual defn of pressure corresponds to box dim — replace $B(x, \varepsilon)$ with $B(x, n, \delta)$, assign weights by φ.

* Pesin & Pitskel' gave defn as dimension — critical pt:

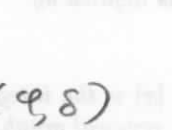
$$\mathcal{P}(Z, N, \delta) = \{\{(x_i, n_i)\}_{i \in I} \mid x_i \in Z,\ n_i \geq N,\ \bigcup_i B(x_i, n_i, \delta) \supset Z\}$$

$$m_P(Z, \varphi, t, N, \delta) = \inf_{\mathcal{P}(Z,N,\delta)} \sum_{i \in I} e^{-n_i t + S_{n_i}\varphi(x_i)}$$

$$m_P(Z, \varphi, t, \delta) = \lim_{N \to \infty} m_P(Z, \varphi, t, N, \delta)$$

$$P_Z(\varphi, \delta) = \text{crit } t, \qquad P_Z(\varphi) = \lim_{\delta \to 0} P_Z(\varphi, \delta)$$

* Barreira & Schmeling (2000) used this to remove requirement ② above.

* What if uniform expansion fails? eg. ,

Z as before. Still have $\lambda(x) > 0$ on Z...

Thm (C., 2009)

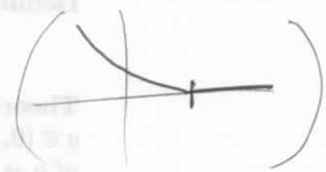
X a cpt met sp, $f: X \circlearrowleft$ conformal, no crit pts

$\left(a(x) = \lim_{y \to x} \frac{d(fx, fy)}{d(x,y)} \text{ exists \& is cts, nonzero} \right)$

Defn $\underline{\lambda}(x) = \liminf_{n \to \infty} \frac{1}{n} S_n(\log a)(x), \; \overline{\lambda}(x)$

Let $Z \subset X$ be such that for every $x \in Z$,

1. $0 < \underline{\lambda}(x) \leq \overline{\lambda}(x) < \infty$
2. Either (a) $\underline{\lambda}(x) = \overline{\lambda}(x)$ or (b) $\inf \{ S_n(\log a)(f^k x) \mid n,k \} > -\infty$.



Then $\dim_H Z = \inf \{ t \mid P_Z(-t \log a) \leq 0 \}$
$= \sup \{ t \mid P_Z(-t \log a) > 0 \}$

Remark. Suppose $\lambda(x) = \lim_{n \to \infty} \frac{1}{n} S_n(\log a) = \lambda \in \mathbb{R}^+ \;\; \forall x \in Z$.

Then one may show that $P_Z(-t \log a) = P_Z(0) - t\lambda$

and since $P_Z(0) = h_{top} Z$, we have $\boxed{\dim_H Z = \frac{h_{top} Z}{\lambda}}$

* For an application of this, we turn to multifractal analysis.

§2 Birkhoff Ergodic Thm

μ ergodic, $\varphi \in L^1(\mu) \Rightarrow \frac{1}{n} S_n \varphi(x) \longrightarrow \int \varphi \, d\mu \quad \mu$-a.e.

BUT, of course, the limit may take many different values, and different measures see different limits.

Ex1 $\varphi = \chi_I$, in previous example, $S_n \varphi(x) = a_n(x)$

Easy to construct pts with any limit in $[0,1]$

Defn $K_\alpha^\varphi = \{ x \mid \frac{1}{n} S_n \varphi(x) \longrightarrow \alpha \}$

$X = \left(\bigcup_{\alpha \in \mathbb{R}} K_\alpha^\varphi \right) \cup X'$ multifractal decomposition

Q How big are the various level sets?

* How to quantify K^φ_α? Measures useless. So use a dimensional quantity — but they are dense, so can't use box dim. Thus, use $\dim_H$, or h_{top}.

Defn The Birkhoff spectrum of φ is

$$B(\alpha) = h_{top} K^\varphi_\alpha$$

Multifractal miracle $B(\alpha)$ is often smooth & concave!

Why? Legendre transform:

$$(1) \qquad B(\alpha) = \inf_{t \in \mathbb{R}} \left(P_X(t\varphi) - t\alpha \right)$$

$$(2) \qquad P_X(t\varphi) = \sup_{\alpha \in \mathbb{R}} \left(B(\alpha) + t\alpha \right)$$

* If (1) & $t \mapsto P_X(t\varphi)$ smooth & strictly convex, then $B(\alpha)$ smooth & strictly concave.

* To get (1), first get (2) !

2 incorrect arguments

I

$$P_X(t\varphi) \boxed{=} \sup_{\alpha \in \mathbb{R}} P_{K^\varphi_\alpha}(t\varphi) = \sup_{\alpha \in \mathbb{R}} \left(P_{K^\varphi_\alpha}(0) + t\alpha \right)$$

$$= \sup_{\alpha \in \mathbb{R}} \left(B(\alpha) + t\alpha \right)$$

II

$$P_X(t\varphi) = \sup_{\mu \in M_e} \left(h_\mu(f) + \int t\varphi \, d\mu \right)$$

$$= \sup_{\alpha \in \mathbb{R}} \sup_{\substack{\mu \in M_e \\ \int \varphi d\mu = \alpha}} \left(h_\mu(f) + t\alpha \right)$$

$$= \sup_{\alpha \in \mathbb{R}} \left[\left(\sup_{\mu(K^\varphi_\alpha) = 1} h_\mu(f) \right) + t\alpha \right]$$

$$\boxed{=} \sup_{\alpha \in \mathbb{R}} \left(h_{top} K^\varphi_\alpha + t\alpha \right) = \sup_{\alpha \in \mathbb{R}} \left(B(\alpha) + t\alpha \right)$$

* ~~Fortunately~~, 2 wrongs make a right — each of these gives one inequality

$\boxed{\text{Thm}}$ (C., 2009)

X cpt, $f: X \circlearrowleft$ cts, $\varphi: X \to \mathbb{R}$ cts.

$$\left.\begin{array}{l} \text{Suppose that for } q \in (q_1, q_2) \text{ we have} \\ \text{I. } \exists \text{ an equilibrium state } \mu_q \text{ for } q\varphi \\ \text{II. } q \mapsto P(q\varphi) \text{ differentiable at } q \\ (\text{or } \mu_q \text{ is unique}) \end{array}\right\} (*)$$

$\boxed{A}$ Then $q \mapsto P(q\varphi)$ is Leg. transform of $B(\alpha)$

$\boxed{B}$ If $(*)$, then let $\alpha_i = \frac{d}{dq} P(q\varphi)\Big|_{q=q_i}$, and

$$B(\alpha) = \inf_{q \in \mathbb{R}} \left(P(q\varphi) - q\alpha \right)$$

$\forall\ \alpha \in (\alpha_1, \alpha_2)$

$\to$ Thus B strictly concave, & C^1 except at α s.t. $P(q\varphi)$ affine.

Special case — $\varphi = \log|Df|$ is geometric potential.

* Get Lyapunov spectrum $h_{top} K_\alpha^{\log|Df|}$
* $\lambda(x)$ constant on $K_\alpha^{\log|Df|}$ $\Rightarrow$ gen. Bowen's eq. applies

if f conformal w/o critical pts:

$$\boxed{\begin{aligned} \dim_H K_\alpha^{\log|Df|} &= \frac{1}{\alpha} h_{top} K_\alpha^{\log|Df|} \\ &= \frac{1}{\alpha} \inf_{q \in \mathbb{R}} \left[P(q \log|Df|) - q\alpha \right] \end{aligned}}$$

Other multifractal spectra:

* 2 ingredients:
  ① local quantity — Birkhoff, Lyapunov, entropy, dimension
  ② dimensional quantifier — entropy or H. dim.
* map

$$\begin{tikzcd} \text{pressure} \arrow[r] & \text{Birkhoff} \arrow[r, "\boxed{\text{Gibbs}}"] \arrow[d] \arrow[r, phantom, shift right=4ex, "\boxed{\text{conformal}}"] & \text{entropy} \arrow[d] \\ & \text{Lyapunov} & \text{dimension} \end{tikzcd}$$