Effective hyperbolicity and SRB measures

Vaughn Climenhaga

University of Houston

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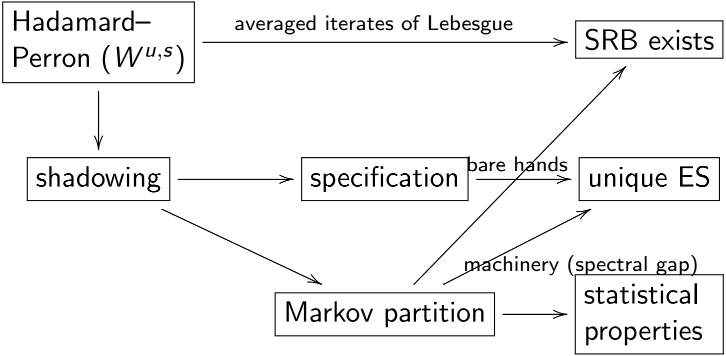
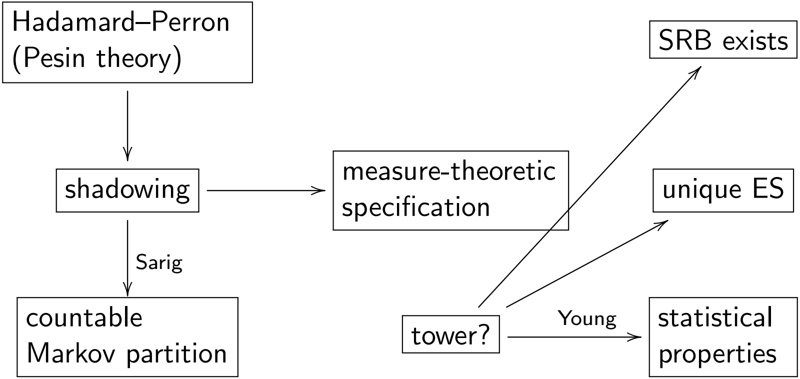
Joint work with Yakov Pesin (PSU)
and Dmitry Dolgopyat (Maryland)

The big picture: uniform hyperbolicity

Goal: Existence, uniqueness, and statistical properties for physical (SRB) measures and equilibrium states for diffeo $f: M \circlearrowleft$

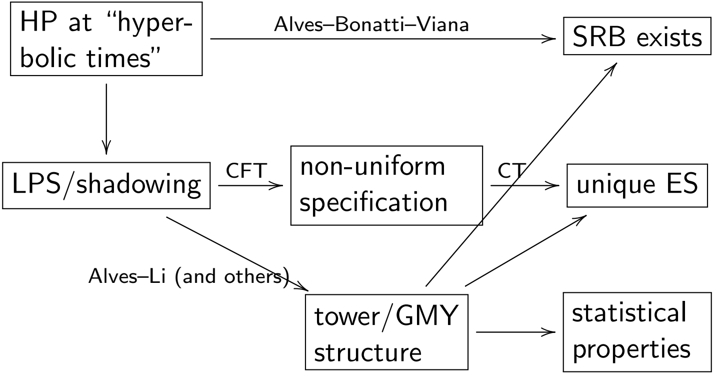


The big picture: non-uniform hyperbolicity

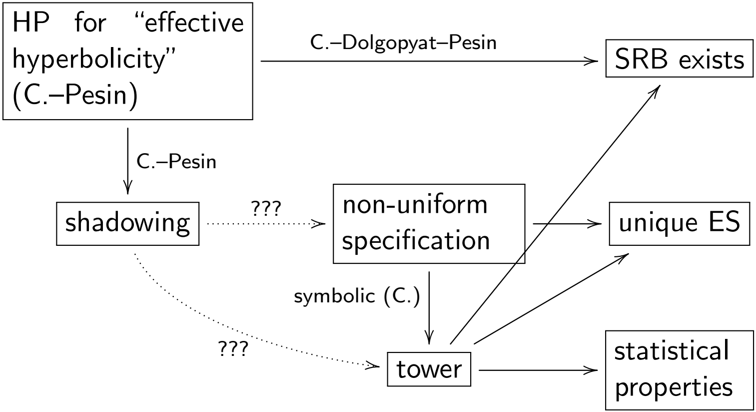


The big picture: dominated splittings

Uniform geometry ($TM = E^1 \oplus E^2$), non-uniform dynamics



The big picture: non-uniform hyperbolicity again



Uniform hyperbolicity

Use local coords around orbit of x, get $f_n \colon \mathbb{R}^d \circlearrowleft$ with $f_n(0) = 0$

- $E_n^{s,u}$ invariant under $Df_n(0)$, uniformly transverse
- $\lambda_n^s := \log \|Df_n(0)|_{E^s}\| < 0 < \lambda_n^u := \log \|Df_n(0)|_{E^u}^{-1}\|^{-1}$
- An admissible manifold of size r is the graph of $\psi_n \colon B(0,r) \cap E_n^u \to E_n^s$ with $\|D\psi_n\| \leq \gamma$.
- **Admissibles stay big:** W_n admissible of size $r \Rightarrow$ some $\hat{W}_n \subset W_n$ has $f_n(\hat{W}_n)$ admissible of size r
- **Admissibles expand:** $x, y \in W_n \Rightarrow d(f_n x, f_n y) > e^\chi d(x,y)$ for $\chi < \left|\lambda_n^{s,u}\right|$.

Recover usual Hadamard–Perron by taking $W_0^u = \lim_{n\to\infty} f_{-n,0}(\hat{W}_{-n})$ where $f_{i,j} = f_{j-1} \circ \cdots \circ f_{i+1} \circ f_i$

Non-uniform hyperbolicity

Now assume $\theta_n = \angle(E^u_n, E^s_n)$ can be arbitrarily small, and $\lambda^{s,u}_n$ can be anything. Hadamard–Perron type results available from Pesin theory, but size of admissible and rate of expansion can decay.

- Size $\approx r/C_n$, backwards contraction by $f^{-1}_{k,n} \approx C_n e^{-(n-k)\chi}$
- C_n depends on asymptotic behaviour of $\lambda^{s,u}_j, \theta_j$

We want a result that depends only on finitely many iterates

Dominated splitting: $\theta_n \gg 0$, $\lambda^s_n < \lambda^u_n$ (but sign can vary)

- n a χ-hyperbolic time if $\sum_{j=k}^{n-1} \lambda^u_j > (n-k)\chi$ for all $0 \leq k < n$
- $f_{0,n}(\hat{W}_0)$ 'big' (size r) at hyperbolic times
- If x, y lie in an admissible $f_{0,n}(W_0)$ and $\left|\lambda^{s,u}_i\right| > \chi$, then $d(f^{-1}_{k,n}x, f^{-1}_{k,n}y) < e^{-(n-k)\chi} d(x,y)$ for all $0 \leq k < n$

Effective hyperbolicity

Now let $\lambda_n^{s,u}, \theta_n$ be arbitrary and assume f_n is C^2. Consider

$$\Delta_n = \max(0,\ \lambda_n^s - \lambda_n^u) \qquad \text{(defect from domination)},$$
$$L = \sup_n |\log(\theta_{n+1}/\theta_n)|.$$

Fix $\bar{\theta} > 0$ and put $\lambda_n^e = \begin{cases} \lambda_n^u - \Delta_n & \theta_n > \bar{\theta}, \\ -L & \text{otherwise.} \end{cases}$

- Effective hyp. time: $\sum_{j=k}^{n-1} \lambda_j^e > (n-k)\chi$ for all $0 \leq k < n$.

Theorem (C.–Pesin)

If n is an effective hyperbolic time for $\{f_j\}$ then $f_{0,n}(\hat{W}_0)$ is large and has uniform backwards contraction for all $f_{k,n}$, $0 \leq k < n$.

Also get control of ‘nearby admissibles’ not passing through 0.

μ an SRB measure for f if

- hyperbolic: all Lyapunov exponents non-zero
- absolutely continuous conditionals on unstable manifolds

SRB measures are physical: describe Lebesgue-typical trajectories

Natural method to build SRB in uniformly hyperbolic setting

- m = Lebesgue measure (volume) on some admissible manifold
- Cesàro averages $\mu_n = \frac{1}{n}\sum_{k=0}^{n-1} f_*^k m$, then $\mu_{n_j} \to \mu$ invariant

Pesin–Sinai, Bonatti–Viana: extends to $E^{cs} \oplus E^u$ if

$$\{x \mid \overline{\lim} \frac{1}{n} \sum_{k=0}^{n-1} \lambda_k^s(x) < 0\}$$

has positive volume. Alves–Bonatti–Viana did the case $E^s \oplus E^{cu}$.

Cannot use Pesin theory to build an SRB measure.

- Start with admissible W, let $W_n = f^n(W)$.
- To work with μ_n must know scale where W_n 'close to unstable', and have contraction so densities behave.
- These are good when C_n is small.
- Need good recurrence properties to $\Lambda_C = \{n \mid C_n \leq C\}$.
- Recurrence properties come from ergodic theory.

For small angles and failure of domination, use effective hyp.

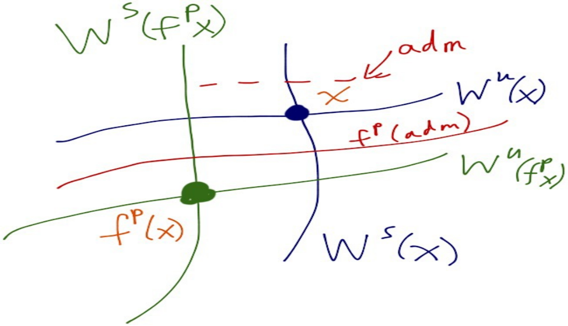
$$\lambda(x) = \min(\lambda^u(x) - \Delta(x), -\lambda^s(x)),$$
$$Q(x, \bar\theta) = \{n \in \mathbb{N} \mid \angle(E^u(f^n x), E^s(f^n x)) < \bar\theta\},$$
$$S = \{x \mid \underline{\lim} \tfrac{1}{n} \textstyle\sum_{k=0}^{n-1} \lambda(f^k x) > 0 \text{ and } \lim_{\bar\theta \to 0} \overline{d}(Q(x, \bar\theta)) = 0\}$$

Theorem (C.–Dolgopyat–Pesin)

If $\mathrm{Leb}(S) > 0$ *then* f *has an SRB measure.*

Closing lemma – uniform hyperbolicity

Orbit segment $x, f(x), \ldots, f^p(x) \approx x$. Periodic point nearby?



f^p induces graph transform on space of u-admissible manifolds

- Contraction $\Rightarrow$ fixed point, similarly for s-admissibles
- Intersection is periodic point

Effective hyperbolicity – explicit constants

Consider finite orbit segment $\{f_n \mid 0 \leq n < p\}$

- $L = \max(|Df_n|_\alpha, \ |\log(\frac{\theta_{n+1}}{\theta_n})|, \ |\log(\frac{\|Df_n(0)(v)\|}{\|v\|})|)$
- $\lambda_n^e = \lambda_n^u - \Delta_n - L\mathbf{1}_{\{\theta_n < \bar{\theta}\}}$
- $M_n^u = \max_{0 \leq m < n} \left((n-m)\chi^u - \sum_{k=m}^{n-1} \lambda_k^e\right)$, similarly M_n^s

Definition

Orbit segment is completely effectively hyperbolic with parameters $M, \theta > 0$ and rates $\chi^s < 0 < \chi^u$ if $\theta_0, \theta_p > \theta$ and

$$M \geq \max(M_p^u, M_p^s, M_0^u, M_0^s),$$

$$M \geq M_n^u + \sum_{k=0}^{n-1}(\lambda_k^s - \chi^s) \text{ for all } 0 \leq n \leq p,$$

and similarly for M_n^s.

Finite-information closing lemma

Theorem (C.–Pesin)

Fix parameters M, θ and rates $\chi^{s,u}$. Given $\delta > 0$ there is $\varepsilon > 0$ and $p_0 \in \mathbb{N}$ such that if

1. *$p \geq p_0$ and $\{x, \dots, f^p(x)\}$ is completely effectively hyperbolic with these parameters and rates;*
2. *$d(x, f^p x) < \varepsilon$, and $E^\sigma \subset K^\sigma(x)$ have $d(Df^p(E^\sigma), E^\sigma) < \varepsilon$,*

then there exists a hyperbolic periodic point $z = f^p z$ such that $d(x, z) < \delta$.