# Thermodynamics with small obstructions

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Joint work with Daniel J. Thompson (Penn State)

# The talk in one slide

specification $\Rightarrow$ intrinsic ergodicity (unique MME)

**Theorem (C.–Thompson)**

*If all obstructions to specification have small entropy, then the MME is unique.*

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specification + Bowen property $\Rightarrow$ unique equilibrium state

**Theorem (C.–Thompson)**

*If all obstructions to specification and the Bowen property have small pressure, then the equilibrium state is unique.*

# Topological pressure

Topological dynamical system:

- $X$ a compact metric space, $f\colon X \to X$ continuous
- $\mathcal{M} = \{$Borel $f$-invariant probability measures on $X\}$

Variational principle 1: $h_{\text{top}}(X, f) = \sup_{\mu \in \mathcal{M}} h_\mu(f)$

Variational principle 2: $P(\varphi) = \sup_{\mu \in \mathcal{M}} \left(h_\mu(f) + \int \varphi \, d\mu\right)$

Maximum achieved by MME/equilibrium state

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## Example

$$X = \Sigma_2^+ = \{0,1\}^{\mathbb{N}} \qquad p, q \in \mathbb{R} \qquad \varphi(x) = p\mathbf{1}_{[0]} + q\mathbf{1}_{[1]}$$

Then $P(\varphi) = \log(e^p + e^q)$ and the unique equilibrium state is $(\alpha, 1-\alpha)$-Bernoulli, where $\alpha = \frac{e^p}{e^p+e^q}$.

When is there a unique equilibrium state?

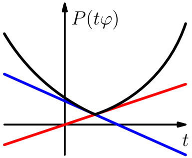
# Thermodynamic formalism

Topological pressure $P(\varphi) = \sup_\mu (h_\mu(f) + \int \varphi \, d\mu)$:

- supremum of affine functions $\Rightarrow$ convex function $C(X) \to \mathbb{R}$;
- equilibrium states are tangent functionals in $C(X)^*$.

If entropy map $\mu \mapsto h_\mu(f)$ is upper semi-continuous, then:

- so is $\mu \mapsto (h_\mu(f) + \int \varphi \, d\mu)$ $\Rightarrow$ *existence*;
- *unique* $\Leftrightarrow$ P (Gâteaux) differentiable;
- convex $\Rightarrow$ diff. on residual set $\Rightarrow$ *unique*.



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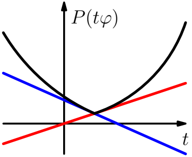



This says nothing about **which** potentials admit unique equilibrium states, or what the ergodic properties of those states are.

**General principle:** Uniform mixing of $X$ and regularity of $\varphi$ should imply uniqueness and strong ergodic properties.

## Thermodynamics for shift spaces

Focus on shift spaces (subshifts):

- $X \subset \Sigma_p^+$ closed and $\sigma$-invariant, NOT NECESSARILY AN SFT
- language of $X$: $\mathcal{L} = \mathcal{L}(X) = \{x_1 \cdots x_n \mid x \in X, n \geq 1\}$
- words of length $n$: $\mathcal{L}_n = \{w \in \mathcal{L} \mid |w| = n\}$
- entropy: $h_{\mathrm{top}}(X, \sigma) = \lim_{n\to\infty} \frac{1}{n} \log \#\mathcal{L}_n$
- pressure: $P(\varphi) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{w\in\mathcal{L}_n} e^{\sup_{x\in[w]} S_n\varphi(x)}$
- $S_n\varphi(x) = \varphi(x) + \varphi(\sigma x) + \cdots + \varphi(\sigma^{n-1}x)$

expansive $\Rightarrow$ entropy map $\mathcal{M} \to \mathbb{R}$ upper semi-continuous

Existence guaranteed: the real question is uniqueness.

Beyond transitivity, what do we need?

# Uniqueness may fail

**Example**

Let $X \subset \Sigma_5^+ = \{0, 1, 2, 1, 2\}^{\mathbb{N}}$ be the shift whose language $\mathcal{L}$ contains $v0^n w$ and $w0^n v$ if and only if $n \geq 2\max(|v|, |w|)$.

- $(X, \sigma)$ is topologically transitive (indeed, mixing)
- $h_{\text{top}}(X, \sigma) = \log 2$
- 2 measures of maximal entropy:

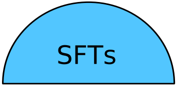
$$\nu = (\tfrac{1}{2}, \tfrac{1}{2})\text{-Bernoulli on } \{1, 2\}^{\mathbb{N}},$$
$$\mu = (\tfrac{1}{2}, \tfrac{1}{2})\text{-Bernoulli on } \{1, 2\}^{\mathbb{N}}.$$

Uniqueness of an MME can fail for transitive shifts.

# Classes of intrinsically ergodic shifts

The following are intrinsically ergodic (unique MME):

- Irreducible subshifts of finite type (Parry 1964)



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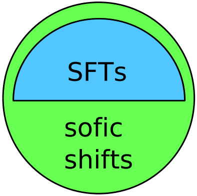

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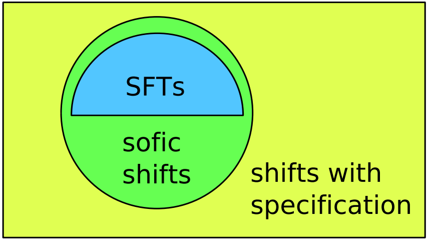

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- Irreducible subshifts of finite type (Parry 1964)
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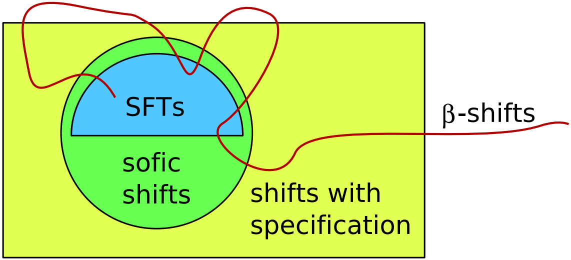
- Shifts with specification (Bowen 1974)
- $\beta$-shifts (Walters 1978, Hofbauer 1979)



## The motivating question

Intrinsic ergodicity is not necessarily preserved by factors.

- $X \subset \{0, 1, 2, 1, 2\}^{\mathbb{N}}$ as before: **two MMEs**
- $Y \subset \Sigma_6^+ = \{0, 1, 2, 1, 2, 3\}^{\mathbb{N}}$: **one MME**
- $\pi \colon Y \to X$ by $\pi(3) = 2$, others unchanged

$$\begin{array}{ccc} Y & \xrightarrow{\sigma} & Y \\ \pi \downarrow & & \downarrow \pi \\ X & \xrightarrow{\sigma} & X \end{array}$$

Specification is preserved by factors, so intrinsic ergodicity survives.

What about $\beta$-shifts? (Klaus Thomsen, Mike Boyle)

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What about $\beta$-shifts? (Klaus Thomsen, Mike Boyle)

**Theorem (C.–Thompson 2010)**

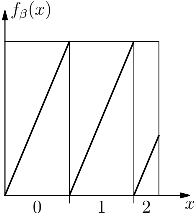
*$(X, \sigma)$ a subshift factor of a $\beta$-shift*
*$\Rightarrow$ all obstructions to specification have zero entropy*
*$\Rightarrow (X, \sigma)$ is intrinsically ergodic*

# $\beta$-shifts

For $\beta > 1$, $\Sigma_\beta$ is the coding space for the map

$$f_\beta \colon [0,1] \to [0,1], \qquad x \mapsto \beta x \pmod 1$$

$1_\beta = a_1 a_2 \cdots$, where $1 = \sum_{n=1}^{\infty} a_n \beta^{-n}$



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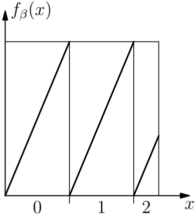


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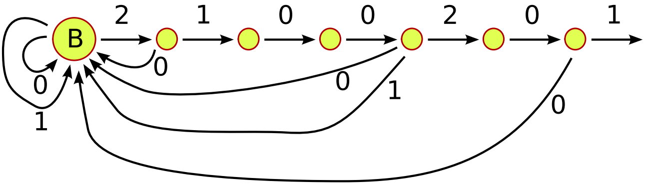
**Fact:** $x \in \Sigma_\beta \Leftrightarrow \sigma^n x \preceq 1_\beta$ for all $n$

$\Leftrightarrow x$ labels a walk starting at **B** on this graph:

(Here $1_\beta = 2100201\ldots$)



# Specification

Topological transitivity $\Rightarrow$ for every $w_1, \ldots, w_m \in \mathcal{L}$ $\exists$ $z_i \in \mathcal{L}$ for which the concatenated word $w_1 z_1 w_2 z_2 \cdots z_{m-1} w_m$ is in $\mathcal{L}$.

## Definition

$X$ has specification if $\exists$ $t \in \mathbb{N}$ such that $z_i$ can always be chosen to have length $t$, independently of $w_i$.

- Mixing SFTs and sofic shifts have specification.
- $\Sigma_\beta$ does not have specification if $1_\beta$ contains arbitrarily long strings of 0's. (Happens for Leb-a.e. $\beta > 1$.)

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The only obstruction to specification is the tail of the sequence $1_\beta$.

**Key idea: zero entropy obstructions are invisible to MMEs**

# Obstructions to specification

$\mathcal{G} \subset \mathcal{L} \rightsquigarrow \mathcal{G}^M := \{vw \mid v \in \mathcal{G}, |w| \leq M\} \rightsquigarrow$ filtration $\mathcal{L} = \bigcup_M \mathcal{G}^M$

**Definition**

$\mathcal{C} \subset \mathcal{L}$ contains all obstructions to specification if $\exists\, \mathcal{G}$ s.t.

- $\mathcal{L} = \mathcal{G}\mathcal{C} := \{vw \in \mathcal{L} \mid v \in \mathcal{G}, w \in \mathcal{C}\}$
- every $\mathcal{G}^M$ has specification ($\mathcal{G}$ is a core for specification)

We can glue words (orbit segments) together, provided we are allowed to remove an obstructing piece from the end of each word.

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## Example

For the $\beta$-shift, take $\mathcal{C} = \{\text{prefixes of } 1_\beta\}$. GRAPH

- $\mathcal{C} =$ words whose path never returns to **B** (cusp excursions)
- $\mathcal{G} =$ words whose path begins and ends at **B**

# Small obstructions

**Theorem (C.–Thompson 2010)**

*If $X$ is a shift space, $\mathcal{C}$ contains all obstructions to specification, and $h(\mathcal{C}) < h_{\mathrm{top}}(X, \sigma)$, then $(X, \sigma)$ is intrinsically ergodic.*

Remark: If the shadowing orbits can be taken to be periodic, then the unique MME is the limit of the measures $\mu_n = \delta_{\mathrm{Per}(n)}$.

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**Proposition**

*$\pi : X \to Y$ a factor map, $\mathcal{C} \subset \mathcal{L}(X)$ contains all obstructions $\Rightarrow$*

- *$\pi(\mathcal{C}) \subset \mathcal{L}(Y)$ also contains all obstructions*
- *$h(\pi(\mathcal{C})) \leq h(\mathcal{C})$*

# Coded systems

A shift space $X$ is coded if its language $\mathcal{L}$ is freely generated by a countable set of generators $\{w_n\}_{n\in\mathbb{N}} \subset \mathcal{L}$.

$$\mathcal{L} = \{\text{all subwords of } w_{n_1} w_{n_2} \cdots w_{n_k} \mid n_i \in \mathbb{N}\}$$

Canonical decomposition $\mathcal{L} = \mathcal{C}^s \mathcal{G} \mathcal{C}^p$ s.t. $\mathcal{G}(M)$ has specification:

$$\mathcal{G} = \{w_{n_1} w_{n_2} \cdots w_{n_k} \mid n_i \in \mathbb{N}\}$$

$$\mathcal{C}^s = \{\text{suffixes of } w_n \mid n \in \mathbb{N}\} \qquad \mathcal{C}^p = \{\text{prefixes of } w_n\}$$

**Obstruction: prefixes and suffixes of generators**

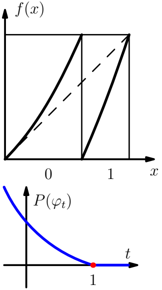
Let $\hat{h} = h(\{\text{prefixes and suffixes of generators}\})$.

- $\hat{h} < h_{\text{top}}(X,\sigma) \Rightarrow (X,\sigma)$ is intrinsically ergodic
- $\hat{h} = 0 \Rightarrow$ every subshift factor of $(X,\sigma)$ is intrinsically ergodic

# The Bowen property

Unique equilibrium states? **Specification is not enough**: consider Manneville–Pomeau map.

- $f(x) = x + x^{1+\varepsilon} \pmod 1$, $\varphi_t = -t \log f'$
- $t < 1$: unique eq. state, fully supported
- $t > 1$: unique eq. state, atomic
- $t = 1$: two equilibrium states



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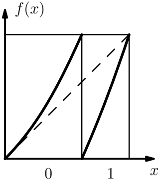


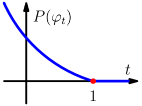


Need control of how $S_n\varphi$ varies across $n$-cylinders.

**Definition**

For $\mathcal{D} \subset \mathcal{L}$, write $V_n(\mathcal{D}, \phi) = \sup_{w \in \mathcal{D}_n} \sup_{x,y \in [w]} |\phi(x) - \phi(y)|$. Then $\varphi$ has the Bowen property on $\mathcal{D}$ if $\sup_n V_n(\mathcal{D}, S_n\varphi) < \infty$.

Hölder potentials on shift spaces have the Bowen property.

# Utility of the Bowen property

Definition of pressure:

$$P(\mathcal{D},\varphi) = \lim_{n\to\infty} \frac{1}{n} \log \Lambda_n(\mathcal{D},\varphi)$$
$$\Lambda_n(\mathcal{D},\varphi) = \sum_{w\in\mathcal{D}_n} e^{\sup_{x\in[w]} S_n\varphi(x)}$$

What happens if we replace $x$ with some other $y \in [w]$?

- Continuity $\Rightarrow$ exponential growth rate $P(\mathcal{D},\varphi)$ preserved
- Bowen $\Rightarrow$ $\Lambda_n$ **preserved to within multiplicative constant**

**Proposition**

*Bowen + specification* $\Rightarrow$ $\frac{1}{C} e^{nP(\varphi)} \leq \Lambda_n(X,\varphi) \leq C e^{nP(\varphi)}$

# Unique equilibrium states

**Theorem (Bowen 1974)**

*Consider a system $(X, f)$ and a potential $\varphi \colon X \to \mathbb{R}$. Suppose*

1. *$X$ a compact metric space;*
2. *$f$ a continuous map;*
3. *$f$ is expansive;*
4. *$f$ has specification;*
5. *$\varphi$ has the Bowen property.*

*Then $\varphi$ has a unique equilibrium state.*

Goal: Replace these with non-uniform versions. Expect to get same result provided obstructions to all properties are small.

# Uniqueness in the presence of obstructions

**Definition**

$\mathcal{C} \subset \mathcal{L}$ contains all obstructions to the Bowen property for $\varphi$ if

- $\varphi$ is Bowen on some $\mathcal{G} \subset \mathcal{L}$ with $\mathcal{L} = \mathcal{G}\mathcal{C}$.

**Theorem (C.–Thompson 2011)**

*Let $X$ be a shift space and $\varphi \in C(X)$. If $\mathcal{C}$ contains all obstructions to specification and the Bowen property for $\varphi$ and $P(\mathcal{C}, \varphi) < P(X, \varphi)$, then there is a unique equilibrium state for $\varphi$.*

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Remark: Each of the following implies that $P(\mathcal{C}, \varphi) < P(X, \varphi)$.

- $h(\mathcal{C}) + \sup_\mu \int \varphi \, d\mu < P(X, \varphi)$
- Every equilibrium state $\mu$ for $\varphi$ has $h(\mu) > h(\mathcal{C})$

# Equilibrium states for $\beta$-shifts

Let $(X, f) = (\Sigma_\beta, \sigma)$ be a $\beta$-shift.

**Theorem (Walters 1978)**

*Every Lipschitz potential $\varphi$ has a unique equilibrium state.*

**Theorem (Hofbauer–Keller 1982)**

*If $\varphi$ has the Bowen property and $\sup \varphi - \inf \varphi < h_{\mathrm{top}}(X, f)$, then $\varphi$ has a unique equilibrium state.*

**Theorem (C.–Thompson 2011)**

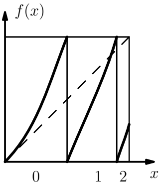
*Every Bowen potential $\varphi$ has a unique equilibrium state.*

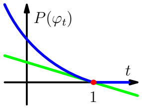
It suffices to prove that every equilibrium state has positive entropy.

# Variants of Manneville–Pomeau

Generalise Manneville–Pomeau:

- $\gamma > 0 \rightsquigarrow f_\gamma(x) = x + \gamma x^{1+\varepsilon} \pmod 1$.
- Coding space is $\Sigma_\beta$ for some $\beta > 1$.
- For most values of $\gamma$, specification fails.
- $\varphi_t = -t \log f'$ is not Bowen.




Remark: For shape of pressure graph, must show

- $\exists \chi > 0$ s.t. $\underline{\lambda}(x) \geq \chi$ for Leb-a.e. $x$;
- $h(\mu) \leq \lambda(\mu)$ for every $\mu \in \mathcal{M}$.

**Theorem (C.–Thompson 2011)**

*For $t < 1$ and $\varepsilon \in (0,1)$, there is a unique equilibrium state for $\varphi_t$.*

# Variations of Manneville–Pomeau (cont.)

$\mathcal{C}$ must capture all obstructions to both specification and the Bowen property.

**Bowen property $\Leftrightarrow$ bounded distortion**

1. Bounded distortion on $\mathcal{D} := \{w \mid w_{|w|} > 0\}$
2. Take $\mathcal{C}$ as before and expand it to
$$\hat{\mathcal{C}} := \{w0^k \mid w \in \mathcal{C} \cup \sigma\mathcal{C}, k \geq 0\}$$
3. $\hat{\mathcal{G}} = \{w \in \mathcal{G} \mid w_{|w|} > 0\} \cup \{wb \mid w \in \mathcal{G}\}$
4. $P(\mathcal{C}, \varphi_t) = \max(\overline{\lim} \frac{1}{n} S_n\varphi_t(x), \varphi_t(0)) = \overline{\lim} \frac{1}{n} S_n\varphi_t(x)$
5. Uses similar argument to Bowen potentials on $\Sigma_\beta$ provided $P(\varphi_t) > 0$.

# Proof sketch for unique MME (Bowen's proof)

Step 1. Constructible MME $\mu$

Step 2. $\mu$ is ergodic and Gibbs.

Step 3. No room for another MME $\nu \perp \mu$.

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# Proof sketch for unique MME (Bowen's proof)

Step 0. Counting estimates. $Ce^{nh} \leq \#\mathcal{D}_n \leq C'e^{nh}$ whenever $\nu(\mathcal{D}_n) \gg 0$ for some MME $\nu$

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$$\mu(\mathcal{D}_n) \geq Ke^{-nh}\#\mathcal{D}_n \geq KC > 0$$

# Proof sketch for unique MME (Our proof)

Step 0. Counting estimates. $Ce^{nh} \leq \#\mathcal{D}_n \leq C'e^{nh}$ whenever $\nu(\mathcal{D}_n) \gg 0$ for some MME $\nu$

Step 1. Constructible MME $\mu = \lim_n N(\sum_{w \in \mathcal{L}_n} \delta_{w,n})$

Step 2. $\mu$ is ergodic and Gibbs. For $w \in \mathcal{L}_n$,

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# Proof sketch for unique MME (Our proof)

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$$\mu((\mathcal{D} \cap \mathcal{G}(M))_n) \geq K(M)e^{-nh} \#(\mathcal{D} \cap \mathcal{G}(M))_n \geq K(M)C(M) > 0$$

# Bowen potentials on $\beta$-shifts

$X$ a $\beta$-shift and $\varphi$ a Bowen potential. $\mathcal{C}$ as before.

- Need to check that $P(\mathcal{C}, \varphi) < P(X, \varphi)$.

For a $\beta$-shift, $\mathcal{C}_n$ contains exactly one word, so $\mathcal{C}$ has zero entropy.

For $\varphi \neq 0$, need to show that $\mathcal{C}$ has small pressure:

$$\varlimsup_{n\to\infty} \frac{1}{n} S_n\varphi(1_\beta) < P(X, \varphi).$$

**Key step:** **exponentially many words near $1_\beta$**

# Estimating $P(\mathcal{C}, \varphi)$, Part I

1. Write $1_\beta = u_1 u_2 u_3 \cdots$, where $u_j = 0^{\ell_j} a_j$ for some $a_j > 0$.
2. Can replace any $u_j$ with $\hat{u}_j = 0^{\ell_j} 0$ and get a legal word.
3. Can change any $k$ of the first $n$ words $u_j$ to $\hat{u}_j$; this produces $\binom{n}{k}$ distinct words $w \in \mathcal{L}_N$, where $N = |u_1 u_2 \cdots u_n|$.
4. Each such word has $\inf_{x \in [w]} S_N \varphi(x) \geq S_N \varphi(1_\beta) - 5kV$, where $V = \sup_{w \in \mathcal{L}} \sup_{x,y \in [w]} |S_{|w|}\varphi(x) - S_{|w|}\varphi(y)|$.
5. $\Lambda_N(\mathcal{L}, \varphi) \geq \sum_{k=0}^{n} \binom{n}{k} e^{S_N \varphi(1_\beta) - 5kV} = e^{S_N \varphi(1_\beta)} (1 + e^{-5V})^n$
6. If $1_\beta$ has a positive frequency of non-zero symbols, then $n \geq \delta N$ for some $\delta > 0$, so this suffices.

# Estimating $P(\mathcal{C}, \varphi)$, Part II

What if there is not a positive frequency of non-zero symbols?

1. $\frac{n}{N} \to 0 \Rightarrow \delta_{1_\beta,n} \xrightarrow{\text{wk}^*} \delta_0 \Rightarrow \frac{1}{n} S_n\varphi(1_\beta) \to \varphi(0)$.
2. Fix $1 < \beta' < \beta$ such that $\Sigma_{\beta'}$ is an SFT.
3. $\varphi|_{\Sigma_{\beta'}}$ has a unique equilibrium state, which is not $\delta_0$:

$$\varphi(0) < P(\Sigma_{\beta'}, \varphi) \leq P(\Sigma_\beta, \varphi).$$

If $\underline{\lim}\ \frac{n}{N} = 0$ and $\overline{\lim}\ \frac{n}{N} > 0$, need to combine both arguments.

Remark: Argument adapts to $f_\gamma$ and $\varphi_t$ provided $P(\varphi_t) > 0$.

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