SRB measures without symbolic dynamics or dominated splittings

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Joint work with Dmitry Dolgopyat and Yakov Pesin

1. Introduction and classical results
   - Definition of SRB measure
   - Examples, known and otherwise
2. General method
   - Decomposing the space of invariant measures
   - Recurrence to compact sets
3. Recurrence to $\mathcal{S}_n(K)$
   - Sequences of local diffeomorphisms
   - Frequency of large admissible manifolds
   - Existence of an SRB measure
4. Applications
   - Maps on the boundary of Axiom A: Slowdown, no shear
   - Maps on the boundary of Axiom A: Slowdown and shear

Physically meaningful invariant measures

- M a compact Riemannian manifold
- $f : M \to M$ a $C^{1+\varepsilon}$ local diffeomorphism
- $\mathcal{M}$ the space of Borel measures on M
- $\mathcal{M}(f) = \{\mu \in \mathcal{M} \mid \mu \text{ is } f\text{-invariant}\}$

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Birkhoff ergodic theorem. If $\mu \in \mathcal{M}(f)$ is ergodic then it describes the statistics of μ-a.e. trajectory of f: for every integrable φ,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(f^k(x)) = \int \varphi \, d\mu$$

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To be "physically meaningful", a measure should describe the statistics of *Lebesgue*-a.e. trajectory.

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"absolutely continuous" $\rightsquigarrow$ "a.c. *on unstable manifolds*"

$\mu \in \mathcal{M}(f)$ is an SRB measure if

1. all Lyapunov exponents non-zero;
2. μ has a.c. conditional measures on unstable manifolds.

SRB measures are physically meaningful. Goal: Prove existence of an SRB measure.

Uniform geometric structure

SRB measures are known to exist in the following settings.

- Uniformly hyperbolic f (Sinai, Ruelle, Bowen)
- Partially hyperbolic f with positive/negative central exponents (Alves–Bonatti–Viana, Burns–Dolgopyat–Pesin–Pollicott)

Key tool is a dominated splitting $T_x M = E^s(x) \oplus E^u(x)$.

1. E^s, E^u depend continuously on x.
2. $\measuredangle(E^s, E^u)$ is bounded away from 0.

Both conditions fail for non-uniformly hyperbolic f.

Non-uniformly hyperbolic maps

The Hénon maps $f_{a,b}(x, y) = (a - x^2 - by, x)$ are a perturbation of the family of logistic maps $g_a(x) = a - x^2$.

1. g_a has an absolutely continuous invariant measure for "many" values of a. (Jakobson)
2. For b small, $f_{a,b}$ has an SRB measure for "many" values of a. (Benedicks–Carleson, Benedicks–Young)
3. Similar results for "rank one attractors" – small perturbations of one-dimensional maps with non-recurrent critical points. (Wang–Young)

Genuine non-uniform hyperbolicity, but only one unstable direction, and stable direction must be strongly contracting.

Other non-uniformly hyperbolic maps

Other examples:

1. Hénon $f_{a,b}(x, y) = (a - x^2 - by, x)$ for $b \gg 0$.
2. Generalised Hénon $f_{a,b}(x, y, z) = (a - y^2 - bz, x, y)$: expect to have two unstable directions, so not rank one.
3. Large perturbations of Axiom A maps: Katok construction (slowdown near hyperbolic fixed point), no dominated splitting; slowdown + shear, no continuous splitting.
4. Small perturbations of maps with SRB measures: either local or global.

Goal: Develop a method for establishing the existence of an SRB measure that can be applied to these and other examples.

Constructing invariant measures

- f acts on $\mathcal{M}$ by $f_*: m \mapsto m \circ f^{-1}$.
- Fixed points of f_* are invariant measures.
- Césaro averages + weak* compactness $\Rightarrow$ invariant measures:

$$\mu_n = \frac{1}{n}\sum_{k=0}^{n-1} f_*^k m \qquad\qquad \mu_{n_j} \to \mu \in \mathcal{M}(f)$$

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Idea: $m =$ volume ⇒ μ is an SRB measure.

$$H = \{x \in M \mid \text{all Lyapunov exponents non-zero at } x\}$$

$$\mathcal{S} = \{\nu \in \mathcal{M} \mid \nu(H) = 1, \nu \text{ a.c. on unstable manifolds}\}$$

- $\mathcal{S} \cap \mathcal{M}(f) = \{\text{SRB measures}\}$
- $\mathcal{S}$ is f_*-invariant, so $m \in \mathcal{S} \Rightarrow \mu_n \in \mathcal{S}$ for all n.
- $\mathcal{S}$ is *not* compact. So why should μ be in $\mathcal{S}$?

Non-uniform hyperbolicity in $\mathcal{M}$

Theme in NUH: choose between invariance and compactness.

Replace unstable manifolds with n-admissible manifolds V:

$d(f^{-k}(x), f^{-k}(y)) \leq Ce^{-\lambda k} d(x,y)$ for all $0 \leq k \leq n$ and $x, y \in V$.

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$\mathcal{S}_n = \{\nu$ supp. on and a.c. on n-admissible manifolds, $\nu(H) = 1\}$.
This set of measures has various non-uniformities.

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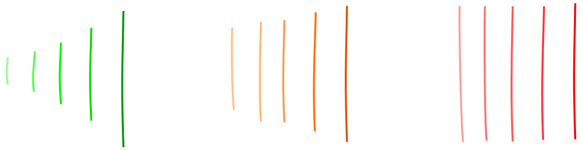

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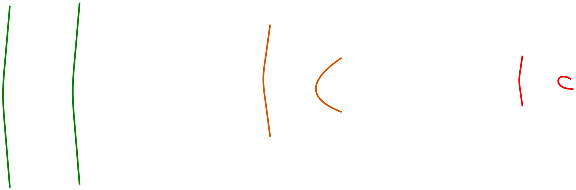

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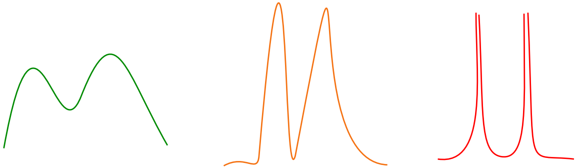

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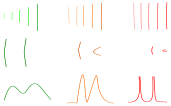

Given $K > 0$, let $\mathcal{S}_n(K)$ be the set of measures for which these non-uniformities are all controlled by K.

$$\text{large } K \Rightarrow \text{worse non-uniformity}$$

$\mathcal{S}_n(K)$ is compact, but not f_*-invariant.

Non-uniformities controlled by K

Admissible manifold V near x defined by

- decomposition $T_x M = G \oplus F$ with $\alpha = \measuredangle(G, F)$,
- $C^{1+\varepsilon}$ function $\psi\colon G \cap B(0, r) \to F$ with $\|D\psi\| \leq \gamma$ and $|D\psi|_\varepsilon \leq \kappa$ such that $V = \exp_x(\operatorname{graph}\psi)$.

Density $\rho \in C^\varepsilon(V)$ and backwards dynamics satisfy

- $L^{-1} \leq \rho(x) \leq L$ and $\|\rho\|_{C^\varepsilon} \leq L$,
- $d(f^{-k}(x), f^{-k}(y)) \leq Ce^{-\lambda k} d(x, y)$.

K controls all the quantities $\alpha, r, \gamma, \kappa$ (geometry of the admissible manifold), L (density function), and C, λ (dynamics).

Conditions for existence of an SRB measure

- M be a compact Riemannian manifold, $U \subset M$ open, $f: U \to M$ a local diffeomorphism with $\overline{f(U)} \subset U$.
- Let μ_n be a sequence of measures whose limit measures are all invariant. (In applications, $\mu_n = \frac{1}{n}\sum_{k=0}^{n-1} f_*^k \,\mathrm{Leb}$.)
- Fix $K > 0$, write $\mu_n = \nu_n + \zeta_n$, where $\nu_n \in \mathcal{S}_n(K)$.

Theorem (C.–Dolgopyat–Pesin 2011)

If $\overline{\lim}_{n\to\infty} \|\nu_n\| > 0$, then some limit measure of $\{\mu_n\}$ has an ergodic component that is an SRB measure for f.

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The question now becomes: How do we obtain recurrence to the set $\mathcal{S}_n(K)$?

Coordinates in TM

We use local coordinates to write the map f along a trajectory as a sequence of local diffeomorphisms.

- $\{f^n(x) \mid n \geq 0\}$ is a trajectory of f
- $U_n \subset T_{f^n(x)}M$ is a neighbourhood of 0 small enough so that the exponential map $\exp_{f^n(x)} \colon U_n \to M$ is injective
- $f_n \colon U_n \to \mathbb{R}^d = T_{f^{n+1}(x)}M$ is the map f in local coordinates

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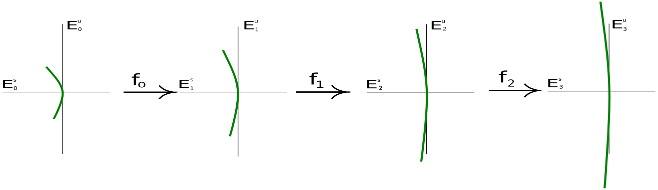
Suppose $\mathbb{R}^d = T_{f^n(x)}M$ has an invariant decomposition $E_n^u \oplus E_n^s$ with asymptotic expansion (contraction) along E_n^u (E_n^s).

$$Df_n(0) = A_n \oplus B_n$$
$$f_n = Df_n(0) + s_n$$
$$f_n(v, w) = (A_n v + g_n(v, w), B_n w + h_n(v, w))$$

Controlling hyperbolicity and regularity

$$\mathbb{R}^d = E_n^u \oplus E_n^s \qquad f_n = (A_n \oplus B_n) + s_n$$

Start with an admissible manifold V_0 tangent to E_0^u at 0, push it forward and define an invariant sequence of admissible manifolds by $V_{n+1} = f_n(V_n)$.



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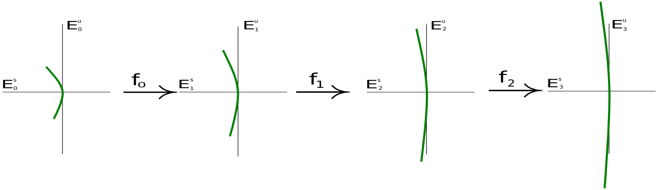
$$V_n = \operatorname{graph} \psi_n = \{v + \psi_n(v)\} \qquad \psi_n \colon B(E_n^u, r_n) \to E_n^s$$

Need to control the size r_n and the regularity $\|D\psi_n\|$, $|\psi_n|_\varepsilon$.



Controlling hyperbolicity and regularity

Consider the following quantities:

$$\lambda_n^u = \log(\|A_n^{-1}\|^{-1}) \qquad \lambda_n^s = \log\|B_n\|$$
$$\alpha_n = \measuredangle(E_n^u, E_n^s) \qquad C_n = \|s_n\|_{C^{1+\varepsilon}}$$

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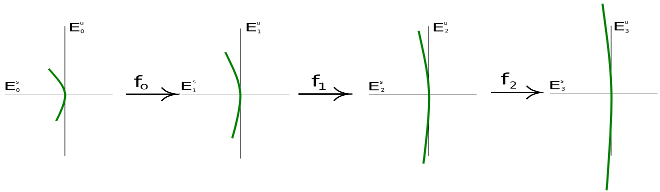
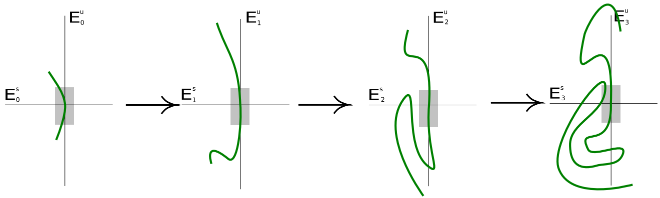


Classical Hadamard–Perron results

Uniform case: Constants such that

- $\lambda_n^s \leq \bar{\lambda}^s < 0 < \bar{\lambda}^u < \lambda_n^u$
- $\alpha_n \geq \bar{\alpha} > 0$
- $C_n \leq \bar{C} < \infty$

Then V_n has uniformly large size: $r_n \geq \bar{r} > 0$.



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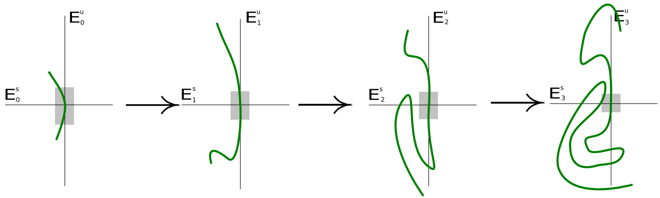
$$C_n \text{ grows slowly} \Rightarrow r_n \text{ decays slowly}$$



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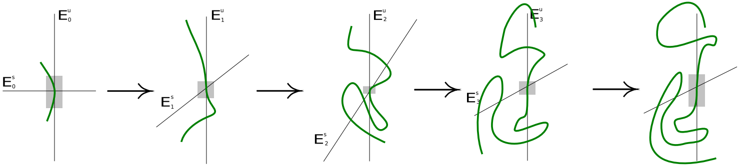
We want to consider the case where

- $\lambda_n^s < 0 < \lambda_n^u$ may fail (may even have $\lambda_n^u < \lambda_n^s$)
- α_n may become arbitrarily small
- C_n may become arbitrarily large (no control on speed)

Usable hyperbolicity

In order to define ψ_{n+1} implicitly, we need control of the regularity of ψ_n. Control $\|D\psi_n\|$ and $|D\psi_n|_\varepsilon$ by decreasing r_n if necessary. So how do we guarantee that r_n becomes "large" again?



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$$\beta_n = C_n(\sin \alpha_{n+1})^{-1}$$

Fix a threshold value $\bar{\beta}$ and define the usable hyperbolicity:

$$\lambda_n = \begin{cases} \min\left(\lambda_n^u,\ \lambda_n^u + \frac{1}{\varepsilon}(\lambda_n^u - \lambda_n^s)\right) & \text{if } \beta_n \leq \bar{\beta}, \\ \min\left(\lambda_n^u,\ \lambda_n^u + \frac{1}{\varepsilon}(\lambda_n^u - \lambda_n^s),\ \frac{1}{\varepsilon}\log\frac{\beta_n}{\beta_{n+1}}\right) & \text{if } \beta_n > \bar{\beta}. \end{cases}$$

A Hadamard–Perron theorem

Write $F_n = f_{n-1} \circ \cdots \circ f_1 \circ f_0 \colon U_0 \to \mathbb{R}^d = T_{f^n(x)}M$. Let $V_0 \subset \mathbb{R}^d$ be a $C^{1+\varepsilon}$ manifold tangent to E_0^u at 0, and let $V_n(r)$ be the connected component of $F_n(V_0) \cap (B(E_n^u, r) \times E_n^s)$ containing 0.

Theorem (C.–Dolgopyat–Pesin 2011)

Suppose $\bar{\beta}$ *and* $\bar{\chi} > 0$ *are such that* $\underline{\lim}_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \lambda_k > \bar{\chi} > 0$. *Then there exist constants* $\bar{\alpha}, \bar{\gamma}, \bar{\kappa}, \bar{r} > 0$ *and a set* $\Gamma \subset \mathbb{N}$ *with positive lower asymptotic frequency such that for every* $n \in \Gamma$,

1. $\measuredangle(E_n^u, E_n^s) \geq \bar{\alpha}$;
2. $V_n(\bar{r})$ *is the graph of a* $C^{1+\varepsilon}$ *function* $\psi_n \colon B_{E_n^u}(\bar{r}) \to E_n^s$ *satisfying* $\|D\psi_n\| \leq \bar{\gamma}$ *and* $|D\psi_n|_\varepsilon \leq \bar{\kappa}$;
3. *if* $F_n(x), F_n(y) \in V_n(\bar{r})$, *then for every* $0 \leq k \leq n$, $\|F_n(x) - F_n(y)\| \geq e^{k\bar{\chi}}\|F_{n-k}(x) - F_{n-k}(y)\|$.

Idea of proof

- Start with V_0, study $V_n = F_n(V_0)$.
- Choose r_n, γ_n, κ_n such that V_n has a piece of size r_n that is (γ_n, κ_n)-admissible.

Can improve γ_n, κ_n at the cost of reducing r_n, or vice versa. Give conditions on "goodness parameters" r_n, γ_n, κ_n; inequalities in terms of λ_n^u, λ_n^s, and β_n.

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- Need to truncate parameters at threshold values $\bar{r}, \bar{\gamma}, \bar{\kappa}$.

positive rate of usable hyperbolicity

$\Rightarrow$ non-thresholded parameters improve asymptotically

$\Rightarrow$ thresholded parameters spend positive proportion of time at threshold

Key consequence

- $\mu_0 = \text{Leb}|_{V_0}$
- $\mu_n = \frac{1}{n}\sum_{k=0}^{n-1} f_*^k \mu_0|_{V_k(r_k)}$ (normalised)
- positive asymptotic rate of usable hyperbolicity
  $\Rightarrow \mu_n \in \mathcal{S}_n(K)$ for positive frequency of times n

Key step for applications: Show that the set of points with positive rate of usable hyperbolicity has positive Lebesgue measure. (Either on M or on V_0.)

Cone families

Return to a local diffeomorphism $f\colon U \to M$. Given $x \in M$, a subspace $E \subset T_x M$, and an angle θ, we have a cone

$$K(x, E, \theta) = \{v \in T_x M \mid \measuredangle(v, E) < \theta\}.$$

If E, θ depend measurably on x, this defines a measurable cone family. Suppose $A \subset U$ has positive Lebesgue measure, is forward invariant, and has two measurable cone families $K^s(x), K^u(x)$ s.t.

1. $\overline{Df(K^u(x))} \subset K^u(f(x))$ for all $x \in A$
2. $\overline{Df^{-1}(K^s(f(x)))} \subset K^s(x)$ for all $x \in f(A)$
3. $T_x M = E^s(x) \oplus E^u(x)$

Usable hyperbolicity (again)

Measurable transverse cone families $K^s(x), K^u(x) \subset T_xM$.

$$\lambda^u(x) = \inf\{\log \|Df(v)\| \mid v \in K^u(x), \|v\| = 1\},$$
$$\lambda^s(x) = \sup\{\log \|Df(v)\| \mid v \in K^s(x), \|v\| = 1\}.$$

Let $\alpha(x) = \measuredangle(K^s(x), K^u(x))$. Fix $\bar{\alpha} > 0$ and consider

$$\zeta(x) = \begin{cases} \frac{1}{\varepsilon} \log \frac{\alpha(f(x))}{\alpha(x)} & \text{if } \alpha(x) < \bar{\alpha}, \\ +\infty & \text{if } \alpha(x) \geq \bar{\alpha}. \end{cases}$$

$$\lambda(x) = \min\left\{\lambda^u(x),\ \lambda^u(x) + \frac{1}{\varepsilon}(\lambda^u(x) - \lambda^s(x)),\ \zeta(x)\right\}$$

An existence result

Consider points with positive asymptotic usable hyperbolicity:

$$S = \left\{ x \in A \,\middle|\, \varliminf_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \lambda(f^k(x)) > 0 \text{ and } \varlimsup_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \lambda^s(f^k(x)) < 0 \right\}$$

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If there exists $\bar{\alpha} > 0$ such that $\operatorname{Leb} S > 0$, then f has a hyperbolic SRB measure supported on Λ.

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Fix $x \in U$. Let V be an embedded submanifold such that $T_x V \subset K^u(x)$, and let m_V be leaf volume on V. Suppose that there exists $\bar{\alpha} > 0$ such that $\varliminf_{r\to 0} m_V(S \cap B(x,r)) > 0$. Then f has a hyperbolic SRB measure supported on Λ.

Large perturbations: an indifferent fixed point

$f : U \to M$ an Axiom A local diffeomorphism, $f(p) = p$.

- f has an SRB measure.
- Small perturbations of f are Axiom A.
- Consider perturbation on boundary of "small".

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Near fixed point, f is time-1 map of $\dot{x} = Ax$, where A has no eigenvalues on imaginary axis. Let $\psi \colon [0,1] \to [0,1]$ be such that

- ψ is C^∞ on $(0,1)$;
- $\psi(0) = 0$; $\psi' > 0$ on $(0, r_0)$; $\psi \equiv 1$ on $[r_0, 1]$;
- $\psi(r) \approx r^\alpha$ near 0, for some $\frac{1}{2} < \alpha < 1$.

Near fixed point, let $g =$ time-1 map for $\dot{x} = \psi(\|x\|^2)Ax$, with $g = f$ outside of $V = B(p, r_0)$.

Theorem (C.–Dolgopyat–Pesin 2011)

g has an SRB measure.

Usable hyperbolicity for g

- If f has a smooth invariant measure μ, then $\psi(\|x\|^2)^{-1}d\mu$ defines a smooth invariant measure for g.
- If the SRB measure for f is not smooth, then the attractor for f is not g-invariant.

f is Axiom A $\Rightarrow f$ has invariant cone families $K^u(x)$ and $K^s(x)$

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- $K^u(x)$ and $K^s(x)$ are g-invariant.
- $\lambda^u(x) \geq 0 \geq \lambda^s(x)$ and $\alpha(x) \gg 0$ for every x.
- $\lambda(x) = \lambda^u(x) \geq \chi > 0$ for every $x \notin V$.

$$\frac{1}{n}\sum_{k=0}^{n-1}\lambda(g^k(x)) \geq \chi \cdot \frac{1}{n}\#\{0 \leq k < n \mid g^k(x) \notin V\}$$

Average sojourn times

- $\tau(x) = \min\{t \mid g^t(x) \notin V\}$
- $G(x) = g^{\tau(x)}(x)$
- $\tau_n(x) = \tau(G^{n-1}(x))$

Claim: there exists $R > 0$ such that $\overline{\lim} \frac{1}{n} \sum_{k=1}^{n} \tau_k(x) \leq R$ for Leb-a.e. x.

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- $\Omega(t_1, \ldots, t_n) = \{x \mid \tau_k(x) = t_k \text{ for } 1 \leq k \leq n\}$
- $\mathrm{Leb}\, \Omega(\vec{t}) \leq C^n \prod_{k=1}^{n} t_k^{-\gamma}$ with $\gamma > 2$
- Model (τ_k) with i.i.d. (T_k) such that $P(T_k = t) = Ct^{-\gamma}$
- Claim holds using fact that $E(T_k) < \infty$

An indifferent fixed point with a shear

$f : U \to M$ an Axiom A diffeomorphism, $f(p) = p$. Near fixed point, f is time-1 map of $\dot{x} = Ax$, where A has no eigenvalues on imaginary axis. Let ψ be as before, so that $\psi(\|x\|^2)Ax$ is the slowed-down vector field.

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Let $N\colon \mathbb{R}^d \to \mathbb{R}^d$ be linear such that

- $N(\mathbb{R}^d) \subset \{0\} \times \mathbb{R}^u \subset \ker N$,

and $\xi\colon [0,1] \to [0,1]$ such that

- ξ is C^∞ on $(0,1)$;
- $\xi(0) = 1$; $\xi \equiv 0$ on $[r_0, 1]$.

Near fixed point, let g = time-1 map for $\dot{x} = (\psi(\|x\|^2)A + \xi(\|x\|^2)N)x$, with $g = f$ outside of $V = B(p, r_0)$.

Theorem (C.–Dolgopyat–Pesin 2011)

g has an SRB measure.

Stable cones for g

Shear $\Rightarrow$ stable cone for f is no longer g-invariant. Need to

1. establish existence of stable invariant cones $K^s(x)$ for g;
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Claim: This boils down to estimating average sojourn times.

- $A = V \setminus g(V)$ (just entered neighbourhood of p)
- $B = g(V) \setminus V$ (just left the neighbourhood of p)
- Let $G \colon A \to B$ and $F \colon B \to A$ be the induced maps

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Need to understand action of DG and DF on the space of s-dimensional subspaces of $\mathbb{R}^d$ transverse to $\mathbb{R}^u \times \{0\}$.

- Identify this space with $(\mathbb{R}^u)^s$
- DG acts as a translation (parabolically)
- DF acts as multiplication (hyperbolically)

Stable cones for g (ctd.)

$$\begin{aligned} \{E \subset \mathbb{R}^d \mid E \text{ transverse to } \mathbb{R}^u \times \{0\}\} &\quad\leftrightarrow\quad (\mathbb{R}^u)^s \\ E \longrightarrow \mathbb{R}^u \times \{0\} &\quad\leftrightarrow\quad \vec{v} \longrightarrow \infty \end{aligned}$$

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does not go to $\mathbb{R}^u \times \{0\}$. Given $\vec{v} = (v_1, \dots, v_s) \in (\mathbb{R}^u)^s$, we have

- $\|DG_x(\vec{v})_j\| \geq \|v_j\| - C\tau(x)$,
- $\|DF_x(\vec{v})_j\| \geq \lambda\|v_j\|$, where $\lambda > 1$.

Usable hyperbolicity

$$R_n(x) := \sum_{k=0}^{\infty} C\lambda^{-k}\tau_{n+k+1}(x)$$

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