

SRB MEASURES AND YOUNG TOWERS FOR SURFACE DIFFEOMORPHISMS

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ABSTRACT. We give geometric conditions that are necessary and sufficient for the existence of Sinai–Ruelle–Bowen (SRB) measures for $C^{1+\alpha}$ surface diffeomorphisms, thus proving a version of the Viana conjecture. As part of our argument we give an original method for constructing first return Young towers, proving that every hyperbolic measure, and in particular every SRB measure, can be lifted to such a tower. This method relies on a new general result on hyperbolic branches and shadowing for pseudo-orbits in nonuniformly hyperbolic sets which is of independent interest.

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Part I. Statements of Results

The purpose of this paper is to study the connection between analytic, geometric, dynamical, and statistical properties of surface diffeomorphisms. In particular, we are interested in the way that certain analytic properties, such as hyperbolicity, imply non-trivial geometric structures which in turn produce non-trivial dynamics and statistical behavior. Although we consider only the two-dimensional case, some of our fundamental results should extend to higher dimensions as well.

In §1 we discuss the general philosophy and theoretical framework of our study, define *Sinai-Ruelle-Bowen (SRB) measure* and recall the *Viana conjecture* on the existence of SRB measures. In §1.7 we state Theorem A which, roughly speaking, says that under some mild recurrence condition,

a fat (nonuniformly) hyperbolic set supports an SRB measure,

thus proving a version of the Viana conjecture in the two-dimensional setting. We note that, unlike any of the previous results in this direction, our assumptions are also *necessary*, thus giving an interesting geometric characterization of SRB measures. We give a more detailed review of existing results in §1.8.

Our construction of the SRB measure uses the technique of *Young towers*, which gives additional information about the geometry and structure of the measure. In §2.3 we state Theorem B which, roughly speaking, says that under some mild recurrence condition, a (nonuniformly) hyperbolic set supports a “topological” Young tower and, more specifically,

a fat (nonuniformly) hyperbolic set supports a Young tower.

This result implies Theorem A but is of independent interest. In §3 we state Theorem C, which says that the assumptions of Theorem A (and of Theorem B) are necessary for the existence of an SRB measure. We also state two corollaries of independent interest: Corollary C.1 says that every SRB measure, and more generally every *hyperbolic* measure, is *liftable* to a topological Young tower, and Corollary C.2 gives conditions under which the decay rate of the tail of the tower can be controlled.

Our construction of a Young tower works in a general setting and differs from other constructions in the literature.¹ The starting point is a measurable subset A of a (non-invariant) “uniformly” hyperbolic set bounded by a *nice domain*. Using an abstract argument we extend A to a *rectangle* Γ – a subset with product structure of local stable and unstable curves – which is maximal in a sense, allowing us to build a tower. The key step in producing Γ is Theorem D in §4, which states that to every *almost return* to A one can associate a *hyperbolic branch*; the total collection of such branches “saturates” A to the desired rectangle Γ . The proof of Theorem D is based on two general results, which we state as Theorems E and F in §5.3 and §5.4 respectively. Theorems D, E, and F are new results in non-uniform hyperbolicity theory of independent interest, with Theorem E providing a new version of Katok’s closing lemma and Theorem F giving a new version of the shadowing property.

¹These typically use specific geometric characteristics of the system under consideration.

In Part I of the paper we state all our results. In Part II we state and prove Theorem E and F which, as mentioned above, are general results in the theory of nonuniform hyperbolicity. Part III is devoted to the proofs of the remaining results in our more specific setting. These results have a clear logical interdependence as follows:

$$E \implies D \implies B \implies A \quad (1) \quad \text{and} \quad F \implies C \implies A \quad (2).$$

More details on organization and the relations between the various results are given at the beginning of Parts II and III. See §1.8 and §2.4 for a discussion of related prior work, especially that of Young [62, 63] and Sarig [56].

1. SRB MEASURES AND THE VIANA CONJECTURE: THEOREM A

Throughout this paper, let M be a surface – by which we mean a compact smooth 2-dimensional Riemannian manifold – and let $f: M \rightarrow M$ be a $C^{1+\alpha}$ diffeomorphism, where $\alpha \in (0, 1]$. Let $d(\cdot, \cdot)$ denote the distance function on M , and let m denote Lebesgue measure on M ; that is, the area form induced by the Riemannian metric. Given a curve $W \subset M$, we write m_W for the one-dimensional Lebesgue measure on W defined by the induced Riemannian metric. By “measurable” we always mean “Borel measurable”.

1.1. Physical measures. The first step in the statistical description of the diffeomorphism f is the notion of the “statistical basin of attraction” of a probability measure μ :

$$\mathcal{B}_\mu := \left\{ x \in M : \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \phi(f^k x) = \int \phi d\mu \text{ for all continuous } \phi: M \rightarrow \mathbb{R} \right\}.$$

Equivalently, \mathcal{B}_μ consists of all points for which $\frac{1}{n} \sum_{k=0}^{n-1} \delta_{f^k(x)}$ converges to μ in the weak* topology, where δ_y is the Dirac delta measure on y . If μ is f -invariant and ergodic, then $\mu(\mathcal{B}_\mu) = 1$ by the Birkhoff ergodic theorem, but since Lebesgue measure is the most natural reference measure, we are most interested in finding μ for which \mathcal{B}_μ is large in the following sense.

Definition 1.1. μ is a *physical measure* for f if $m(\mathcal{B}_\mu) > 0$.

Thus a physical measure is a probability measure which describes the asymptotic statistical behavior of a significant (positive Lebesgue measure) subset of the phase space. Not all dynamical systems admit physical measures,² so it is a basic problem to establish the class of dynamical systems which have physical measures before going on to investigate further question related to the possible number of such measures and their structure and properties.

The simplest example of a physical measure is given by the Dirac-delta measure δ_p at an attracting fixed point p . This easily generalizes to the case when p is an attracting periodic point. At the other extreme, if $\mu \ll m$ is ergodic, then $\mu(\mathcal{B}_\mu) = 1$ gives $m(\mathcal{B}_\mu) > 0$, hence μ is physical. Unfortunately it is relatively rare for such absolutely continuous measures to exist, and thus the problem of the existence of a physical measure is quite non-trivial.

²Consider the identity map, or see [40, p. 140] for a more interesting example.

1.2. Hyperbolic measures. In the 1970's, Sinai, Ruelle, and Bowen established existence, as well as geometric and statistical properties, of physical measures for *uniformly hyperbolic* systems. The theory of *Sinai–Ruelle–Bowen measures*, or *SRB measures*, has since been extended to non-uniform hyperbolicity, in which setting we need the following definition.

Definition 1.2 (Hyperbolic measures and non-zero Lyapunov exponents). An invariant probability measure μ is *hyperbolic* if there exists a set $\Lambda \subseteq M$ with $f(\Lambda) = \Lambda$ and $\mu(\Lambda) = 1$ which has *non-zero Lyapunov exponents*, i.e. there exists a measurable Df -invariant decomposition $T_x M = E_x^s \oplus E_x^u$ such that for every $x \in \Lambda$ and unit vectors $e^s \in E_x^s, e^u \in E_x^u$ we have:

- (1) $\lim_{n \rightarrow \pm\infty} \frac{1}{n} \log \|Df_x^n(e^s)\| =: \lambda_x^s < 0 < \lambda_x^u := \lim_{n \rightarrow \pm\infty} \frac{1}{n} \log \|Df_x^n(e^u)\|;$
- (2) $\lim_{n \rightarrow \pm\infty} \frac{1}{n} \log \angle(E_{f^n(x)}^s, E_{f^n(x)}^u) = 0.$

The heart of this definition is that the *Lyapunov exponents* λ_x^s and λ_x^u are *non-zero* and have *opposite signs*;³ the fact that the limits exist is guaranteed by Oseledets' Multiplicative Ergodic Theorem, which also guarantees that although the angle between the two subspaces is not in general bounded away from zero, it cannot degenerate at an exponential rate along any given orbit, as stated in condition (2). We point out that in the 2-dimensional case the Margulis–Ruelle inequality implies that every measure with positive entropy is hyperbolic.

1.3. Sinai–Ruelle–Bowen measures. A fundamental and crucial property of sets with non-zero Lyapunov exponents is that every point $x \in \Lambda$ has a *local stable curve* V_x^s and a *local unstable curve* V_x^u satisfying certain properties which we describe in Definition 1.9 below.⁴ For the moment we use these curves to give the formal definition of SRB measure.

Definition 1.3 (Fat sets). A set $A \subseteq \Lambda$ is *fat* if

$$(1.1) \quad m\left(\bigcup_{x \in A} V_x^s\right) > 0.$$

Definition 1.4 (SRB measures). An invariant probability measure μ is an *SRB measure* if it is hyperbolic and every set $X \subset \Lambda$ with $\mu(X) = 1$ is fat.⁵

One of the key properties of V_x^s is that $d(f^n(y), f^n(x)) \rightarrow 0$ as $n \rightarrow \infty$ for every $y \in V_x^s$. This implies that if $x \in \mathcal{B}_\mu$ for some measure μ then also $y \in \mathcal{B}_\mu$ for every $y \in V_x^s$. Therefore the fatness condition (1.1) together with Birkhoff's Ergodic Theorem implies that any ergodic SRB measure is a

³If both Lyapunov exponents are negative or both are positive, then it can be shown that the corresponding ergodic component of the measure μ is supported on an attracting or repelling periodic orbit respectively; we exclude this trivial situation.

⁴In fact, the existence of local stable and unstable curves can be proved under weaker conditions than those of non-zero Lyapunov exponents, see Definition 1.5 and Theorem 1.11.

⁵This is not the usual definition of SRB measure, but equivalent to it [59, Theorem C].

physical measure.⁶ In his plenary lecture at the ICM in Berlin in 1998, Viana formulated the following natural conjecture.

Conjecture (Viana [60]). *If a smooth map has only non-zero Lyapunov exponents at Lebesgue almost every point, then it admits some SRB measure.*

In this paper we prove a version of the Viana conjecture for surface diffeomorphisms under the hyperbolicity conditions **(H1)**, **(H2)**, and **(H3)** (see Definition 1.5 below; these conditions are weaker than the non-zero Lyapunov exponents condition in Definition 1.4) but with the addition of a mild *recurrence* condition, see Theorem A. It seems likely that any form of Viana's conjecture which can actually be proved would require some recurrence condition. In the rest of this section we give the exact definitions we need for the formal statements of our result and discuss the background literature on the topic.

1.4. Hyperbolic sets. The requirement that the limits in Definition 1.4 exist is somewhat unnatural in a setting where we do not *a priori* have an invariant probability measure, and also obscures a crucial feature of sets with non-zero Lyapunov exponents, which is the fact that the convergence to the limit can be very *non-uniform*. We give here an alternative formulation of hyperbolicity, which is slightly more technical, but is more general and explicit about the intrinsic non-uniformity.

Definition 1.5 (Hyperbolic set). Given $\chi, \epsilon > 0$, we say that an f -invariant measurable set Λ is (χ, ϵ) -*hyperbolic* if there exists a measurable Df -invariant splitting $T_x M = E_x^s \oplus E_x^u$ for all $x \in \Lambda$, and measurable positive functions $C, K: \Lambda \rightarrow (0, \infty)$ satisfying

$$\text{(H1)} \quad e^{-\epsilon} \leq K(f(x))/K(x) \leq e^\epsilon \quad \text{and} \quad e^{-\epsilon} \leq C(f(x))/C(x) \leq e^\epsilon$$

such that for every $x \in \Lambda$

$$\text{(H2)} \quad \angle(E_x^s, E_x^u) \geq K(x)$$

and for all unit vectors $e_x^s \in E_x^s, e_x^u \in E_x^u$ and for all $n \geq 1$,

$$\text{(H3)} \quad \begin{aligned} \|Df_x^n(e_x^s)\| &\leq C(x)e^{-\chi n}, & \|Df_x^n(e_x^u)\| &\geq C(x)^{-1}e^{\chi n}, \\ \|Df_x^{-n}(e_x^s)\| &\geq C(x)^{-1}e^{\chi n}, & \|Df_x^{-n}(e_x^u)\| &\leq C(x)e^{-\chi n}. \end{aligned}$$

A set Λ is χ -*hyperbolic* if it is (χ, ϵ) -hyperbolic for all $\epsilon > 0$, and *hyperbolic* if it is a union of χ -hyperbolic sets over all $\chi > 0$.

We will always assume that both E_x^s and E_x^u are non-trivial (hence one-dimensional) and we stress that our definition of hyperbolicity is inherently *non-uniform* and the set Λ is not in general closed. Moreover, observe that if Λ is (χ, ϵ') -hyperbolic for some $0 < \epsilon' < \epsilon$, then it is (χ, ϵ) -hyperbolic.

⁶ The converse is not true: for example, if p is a hyperbolic fixed point whose stable and unstable curves form a figure-eight, then δ_p is a hyperbolic physical measure which is not SRB [40, p. 140].

Remark 1.6. It can be shown that a set Λ with *non-zero Lyapunov exponents* (as in Definition 1.2) is *hyperbolic* (as in Definition 1.5). Indeed, if Λ has non-zero Lyapunov exponents then it is a union of f -invariant sets on which the Lyapunov exponents λ^s, λ^u are uniformly bounded away from 0. Each such set is then (χ, ϵ) -hyperbolic for some $\chi > 0$ and for *every* $\epsilon > 0$, where the functions $K = K_\epsilon$ and $C = C_\epsilon$ clearly depend on ϵ , see [13, §3.3].⁷

A first advantage of formulating hyperbolicity as above is that we can write Λ as a union of nested sets on which we have uniform estimates.

Definition 1.7 (Regular sets). Given a (χ, ϵ) -hyperbolic set Λ , for each $\ell \geq 1$ we define the *regular level set*

$$(1.2) \quad \Lambda_\ell := \{x \in \Lambda : C(x) \leq e^{\epsilon\ell} \text{ and } K(x) \geq e^{-\epsilon\ell}\}.$$

Clearly $\Lambda_\ell \subseteq \Lambda_{\ell+1} \subseteq \dots$ and $\Lambda = \bigcup_{\ell \geq 1} \Lambda_\ell$ and, by **(H1)**,

$$(1.3) \quad f^{\pm k}(\Lambda_\ell) \subseteq \Lambda_{\ell+k} \quad \text{for all } \ell, k \in \mathbb{N}.$$

Note that for a χ -hyperbolic set Λ the regular sets Λ_ℓ are not defined until we fix a value of $\epsilon > 0$; changing the value of ϵ changes the functions C, K and hence changes the sets Λ_ℓ . On the other hand, we can introduce a useful notation for regular sets independent of an a-priori choice of Λ .

Definition 1.8 ((χ, ϵ, ℓ) -regular sets). A set $\Gamma \subset M$ is (χ, ϵ, ℓ) -*regular* if there exists a (χ, ϵ) -hyperbolic set Λ such that $\Gamma \subset \Lambda_\ell$.

1.5. Local stable and unstable curves. One of the fundamental consequences of hyperbolicity is the existence of stable and unstable curves.

Definition 1.9 (Local stable and unstable curves). For $C, \lambda, r > 0$ and $x \in M$, a C^1 curve V_x^s containing x is called a (C, λ, r) -*local stable curve*, or simply a *local stable curve*, of x if:

- (1) there is a splitting $T_x M = E^u \oplus E^s$ and a C^1 function

$$\psi_x^s : B^s(0, r) := B(0, r) \cap E^s \rightarrow E^u$$

such that $V_x^s = \exp_x\{v + \psi_x^s(v) : v \in B^s(0, r)\}$;

- (2) for every $y, z \in V_x^s$ and $n \geq 1$, $d(f^n(y), f^n(z)) \leq C e^{-\lambda n} d(y, z)$.

A (C, λ, r) -*local unstable curve* V_x^u is defined similarly, interchanging s, u in the first condition and writing $d(f^{-n}(y), f^{-n}(z)) \leq C e^{-\lambda n} d(y, z)$ in the second.

Definition 1.10 (Bracket of local stable and unstable curves). If V_x^s, V_y^u are local stable and unstable curves of x, y respectively, the *bracket of x and y* is

$$[x, y] := V_x^s \cap V_y^u.$$

⁷The converse is not true; the limits in the definition of non-zero Lyapunov exponents need not exist at every point (only almost every), even in uniform hyperbolicity. Although existence of these limits is not necessary for our results, the slow variation condition **(H1)** still plays a crucial role in Theorem 1.11, and it seems unlikely that it can be removed.

Stable and unstable curves V_x^s and V_y^u need not exist for all x, y , and even if they do exist, the bracket $[x, y]$ may be empty. The following classical result gives conditions which guarantee that the curves exist and the bracket is non-empty; see [13, §7.1] or [48] for a more precise and technical statement and the proof.

Theorem 1.11 (Local Stable and Unstable Manifold theorem). *For every $\chi > \lambda > 0$ there exists $\epsilon_0 > 0$ such that, for any $\epsilon \in (0, \epsilon_0)$ and any $\ell \in \mathbb{N}$ there exist constants $C_\ell, r_\ell, \delta_\ell > 0$ such that if Γ is a (χ, ϵ, ℓ) -regular set, then*

- (1) *for every $x \in \Gamma$, the $(C_\ell, \lambda, r_\ell)$ -local stable and unstable curves exist and depend continuously in the C^1 topology on $x \in \Gamma$;*
- (2) *If $x, y \in \Gamma$ satisfy $d(x, y) < \delta_\ell$, then the bracket $[x, y]$ is non-empty and consists of a single point.*

Remark 1.12. The bracket $[x, y]$ of two points belonging to a (χ, ϵ, ℓ) -regular set Γ need not be contained in Γ , or even in the (χ, ϵ) -hyperbolic set Λ which contains Γ . For example, this occurs when Γ consists of two hyperbolic periodic points x, y that are close enough to each other. Theorem F and Corollary 5.9 demonstrate, however, that $[x, y]$ always belongs to *some* hyperbolic set, and indeed more precisely to a $(\lambda/4, 2\epsilon, \ell + \ell')$ -regular set, where ℓ' depends only on χ, λ, ϵ .

Brackets play a very important role in describing the geometry of hyperbolic sets. In particular we will be interested in sets which are *closed* with respect to the bracket operation.

Definition 1.13 (Rectangles). A *rectangle* is a (χ, ϵ, ℓ) -regular set Γ such that for every $x, y \in \Gamma$ we have $[x, y] \in \Gamma$.

Remark 1.14. Rectangles are also sometimes referred to in the literature as sets with *local product structure* or *hyperbolic product structure*, though this is usually in the more restrictive uniformly hyperbolic setting. The discussion above shows that rectangles are very natural structures also in our more general (nonuniformly) hyperbolic setting. Indeed, if A is a (χ, ϵ, ℓ) -regular set of sufficiently small diameter then the bracket $[x, y]$ is well defined and consists of a single point for every pair of points $x, y \in A$. In this case then $[[x, y], [x', y']] = [x, y'] \in \Gamma$ for all $x, y, x', y' \in A$ and therefore the set

$$(1.4) \quad \Gamma := \{[x, y] : x, y \in A\}$$

is closed under the bracket operation. Moreover, as mentioned above, Theorem F and Corollary 5.9 show that Γ is a $(\lambda/4, 2\epsilon, \ell + \ell')$ -regular set, and therefore Γ is a rectangle.

1.6. Definition of constants. Before proceeding further, we give an explicit bound on how small ϵ must be in the (χ, ϵ) -hyperbolic sets we consider. The precise form of this bound is technical and can be omitted at a first reading; the important thing is that as with ϵ_0 in Theorem 1.11, the quantity ϵ_1 depends only on f, χ , and λ . Fix constants $c_1 < 0 < c_2, c_3$ such that

$$(1.5) \quad c_1 = -c_2 \leq \min_{x \in M} \{\log \|D_x f^{-1}\|^{-1}\} \leq \max_{x \in M} \{\log \|D_x f\|\} \leq c_2 < \frac{c_3}{1 + \alpha}.$$

Given $\chi > \lambda > 0$, let ϵ_0 be as in Theorem 1.11. Define the following auxiliary constants:

$$(1.6) \quad \begin{aligned} \gamma &:= \frac{\chi - c_1}{2\chi}, & \beta &:= \frac{2\chi}{c_3 + \chi}\alpha, & \iota &:= \frac{2(\chi - \lambda)}{6\epsilon_0\gamma\alpha + (2 + \alpha\beta)c_2 + 2\chi}, \\ \eta &:= 6\gamma\alpha\iota + 2, & \zeta &:= \alpha\beta\iota. \end{aligned}$$

Notice that $\gamma, \eta > 1$ and $\beta, \iota, \zeta \in (0, 1)$. Let

$$(1.7) \quad \epsilon_1 = \epsilon_1(f, \chi, \lambda) := \min \left\{ \frac{\lambda\alpha}{18}, \frac{\lambda\beta}{7\gamma}, \frac{\lambda\zeta}{\eta - 1}, \frac{\lambda}{2(1 + 1/\alpha)}, \epsilon_0 \right\}.$$

Throughout the paper, we will consider (χ, ϵ) -hyperbolic sets where $\epsilon \in (0, \epsilon_1)$ and ϵ_1 is given by (1.7).

Remark 1.15. The first bound in (1.7) is used in (5.9), the others are used in the proof of Theorem H in §8, when we produce a constant $\delta > 0$ such that for every $\ell \in \mathbb{N}$ and every $x, y \in \Lambda_\ell$ with $d(x, y) \leq \delta e^{-\lambda\ell}$, the corresponding *Lyapunov charts* (§5.1) are *overlapping* (Definition 8.1). More precisely, the second and third bound in (1.7) are used in Lemmas 8.12 and 8.13, and the fourth is used in (8.45). The last bound, $\epsilon_1 \leq \epsilon_0$ is used to guarantee that we can apply the Local Stable and Unstable Manifold Theorem 1.11.

Remark 1.16. The precise value of $\lambda \in (0, \chi)$ is not important and the reader wishing to reduce the number of constants may as well consider $\lambda = \chi/2$, although choosing a different value of λ might yield a larger value of ϵ_1 .

1.7. Nice domains, recurrent sets, and SRB measures. We are now ready to introduce the key definitions we need to state our main result. To simplify the notation, for a positive integer T we let $T\mathbb{N}$ denote the set of positive integer multiples of T . We also use the notation $V_{p/q}^{s/u}$ to refer simultaneously to $V_p^s, V_q^s, V_p^u, V_q^u$.

Definition 1.17 (Nice Domain). Given $\chi, \epsilon > 0$, $\ell \in \mathbb{N}$, and $r \in (0, \delta_\ell)$, a *nice domain* is a topological disk Γ_{pq} whose boundary is formed by (pieces of) the local stable and unstable curves of (χ, ϵ, ℓ) -regular periodic points p, q satisfying $d(p, q) < r$; see Figure 1.1. We let $T = T(\Gamma_{pq})$ denote the minimum common multiple of the periods of p, q , so that $f^T(p) = p$ and $f^T(q) = q$.

Remark 1.18. The “niceness” condition implies that for all $n \in T\mathbb{N}$,

$$(1.8) \quad f^n(V_{p/q}^s \cap \Gamma_{pq}) \cap \text{Int } \Gamma_{pq} = \emptyset \quad \text{and} \quad f^{-n}(V_{p/q}^u \cap \Gamma_{pq}) \cap \text{Int } \Gamma_{pq} = \emptyset;$$

that is, the stable (resp., unstable) boundary of Γ_{pq} never intersects the interior of Γ_{pq} at iterates which are multiples of T in forward (resp., backward) time. This can be thought of as a two-dimensional version of the notion of “nice interval” in one-dimensional dynamics, which refers to an interval whose boundary points never enter the interval in forward time. Here (1.8) will play a crucial role by ensuring that certain regions are necessarily nested or disjoint, see e.g. Remark 4.9 and Lemma 11.9. There is no obvious generalization of

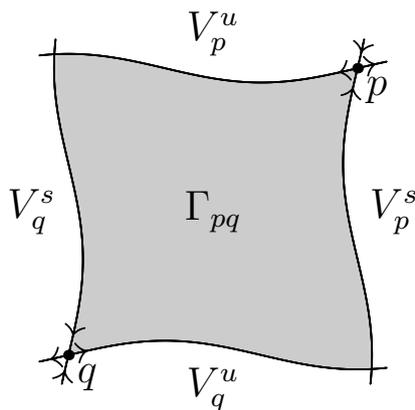


FIGURE 1.1. A nice domain.

this condition to higher dimensions and this is essentially the main reason for which our main results are restricted to surface diffeomorphisms.

Definition 1.19 (Nice set). Given $\chi, \epsilon > 0$, $\ell \in \mathbb{N}$, and $r \in (0, \delta_\ell)$, a set A is *nice* if it is (χ, ϵ, ℓ) -regular and is contained in some nice domain Γ_{pq} .

Remark 1.20. The definitions of nice domains and nice sets depend on constants χ, ϵ, ℓ, r which, for simplicity, are not reflected in the notation. We stress however that any reference to nice domains or nice sets which does not explicitly refer to a choice of constants means that such a choice is implicitly clear from the context. The same applies to the constant T which is always associated to a nice domain and therefore to a nice set (as for example in the following definition in which T is implicitly given by the choice of a nice set A).

Definition 1.21 (Recurrence). A nice set $A \subseteq \Lambda$ is

- (1) *recurrent* if for all $x \in A$ there exist $i, j \in T\mathbb{N}$ such that

$$f^i(x), f^{-j}(x) \in A;$$

- (2) *strongly recurrent* if for all $x \in A$,

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \#\{i \in T\mathbb{N}, 1 \leq i \leq n : f^i(x) \in A\} > 0,$$

and similarly with f^i replaced by f^{-i} .

We are now ready to state our main result, which uses the notion of nice set and of recurrent set as well as the notion of fat set given in (1.1).

Theorem A. *Let f be a $C^{1+\alpha}$ surface diffeomorphism.*

- (1) *For every $\chi > \lambda > 0$, $0 < \epsilon < \epsilon_1(f, \chi, \lambda)$, and $\ell \in \mathbb{N}$, there exists $r > 0$ such that if there exists a nice strongly recurrent fat set A then f admits an SRB measure.*
- (2) *Conversely, if f admits an SRB measure, then for every sufficiently small $\chi > 0$ and every $\epsilon > 0$, there exists $\ell \in \mathbb{N}$ such that for every $r > 0$, there exists a nice strongly recurrent fat set A .*

In addition to the explicit bounds in (1.7) on ϵ , we refer to equation (10.1) for relatively explicit conditions on r .

1.8. Historical background. We review here some of the main results on the existence of SRB measures for diffeomorphisms. To avoid getting too technical we will not be overly specific about the precise technical assumptions, emphasizing instead the general ideas. We refer the reader to [27] for more details and a discussion of the various techniques which have been used in different settings. Most results mentioned below hold in arbitrary dimension.

1.8.1. *Uniformly hyperbolic sets.* In the 1970's, Sinai, Ruelle, and Bowen constructed (in fact *invented!*) SRB measures for *fat, uniformly hyperbolic* sets Λ (attractors) under the additional assumption that Λ has a dense set of periodic points (*Axiom A*) or, equivalently, that Λ has local product structure or that Λ is locally maximal (see [58] for a proof that these three properties of uniformly hyperbolic sets are equivalent).

1.8.2. *Partially hyperbolic attractors.* In 1982, Pesin and Sinai [51] developed a new “push-forward” technique for constructing what they called “ u -Gibbs” measures (also called simply u -measures), which share a lot of geometric characteristics of SRB measures but are not necessarily physical measures. They applied their construction to *partially hyperbolic* attractors. In 2008, Burns, Dolgopyat, Pesin, and Pollicott [20] showed that under some “transitivity” assumptions, a u -measure that has negative Lyapunov exponents for vectors in the central direction on a set of positive measure is a unique SRB measure. In 2000, Bonatti and Viana [16] considered a variation of this setting with a continuous splitting $E^s \oplus E^{uu}$ with uniform expansion estimates in E^{uu} and nonuniform contraction estimates in E^s , and proved the existence of genuine SRB measures.

Around the same time, Alves, Bonatti, and Viana [1] considered the more difficult setting of a continuous splitting $E^{ss} \oplus E^u$ with uniform contraction and nonuniform expansion estimates. The construction of the SRB measure in this case required a significantly more sophisticated version of the push-forward technique. An alternative construction of the SRB measure using Young towers was carried out more recently in [5] under some slightly weaker expansivity assumptions.

1.8.3. *Nonuniformly hyperbolic sets.* Relaxing the continuity of the splitting $E^s \oplus E^u$ and the uniform lower bound on the angle $\angle(E_x^s, E_x^u)$ seems to bring the level of difficulty of the problem to another level. The first result for a system with this kind of hyperbolic set is due to Benedicks and Young who constructed SRB measures for certain two-dimensional “Hénon” maps, first using the push-forward technique [14] and then using Young towers [62, 15], in both cases taking significant advantage of the specific geometric and analytic properties of the maps. These results were extended to more general “Hénon-like” maps in [61] but still only apply to some quite restrictive classes of systems.

More recently, [24] significantly generalizes the techniques of [1] to successfully construct SRB measures for systems in which the splitting $E^s \oplus E^u$ is only measurable, with angle $\angle(E_x^s, E_x^u)$ not bounded away from zero, and with non-uniform contraction and expansion estimates which, however, need to satisfy a non-trivial “synchronization” assumption.

We conclude this review by emphasizing that these prior results all established existence of SRB measures under certain conditions that were proved to be sufficient; however, we are not aware of any previous results demonstrating that any such set of conditions is also *necessary* for existence of an SRB measure.⁸ The conditions in Theorem A above are both necessary and sufficient, so that this is in some sense an optimal result.

2. YOUNG TOWERS: THEOREM B

Our strategy for producing the SRB measure in Theorem A consists of a new technique for the construction of a *Young tower*. The latter has a non-trivial geometric and dynamical structure and it is rather remarkable that its existence can be deduced using only a nice, fat, strongly recurrent set.

2.1. Topological Young Towers. We first recall some standard and some slightly non-standard definitions.

Definition 2.1 (Nice Rectangles). A rectangle Γ is a *nice rectangle* if $\Gamma \subset \Gamma_{pq}$ for some nice domain Γ_{pq} and $p, q \in \Gamma$.

Notice that the above definition is simply saying that Γ is simultaneously a rectangle and a nice set, and specifying additionally that the points p, q which define the nice domain also belong to Γ . If Γ is a nice rectangle, then the stable and unstable curves of every point $x \in \Gamma$ intersect those of p, q since the definition of nice domain requires that $d(p, q) < \delta_\ell$. In other words, they “cross” the domain completely, either in the stable or in the unstable direction. We then want to restrict our attention to the pieces of local stable and unstable curves which are inside the nice domain and so, for any $x \in \Gamma$ (including the points p, q), let

$$W_x^{s/u} := V_x^{s/u} \cap \Gamma_{pq}.$$

Then a nice rectangle $\Gamma \subset \Gamma_{pq}$ has the structure

$$(2.1) \quad \Gamma = C^s \cap C^u \quad \text{where} \quad C^s := \bigcup_{x \in \Gamma} W_x^s \quad \text{and} \quad C^u := \bigcup_{x \in \Gamma} W_x^u.$$

Definition 2.2 (*s*-subsets and *u*-subsets). If Γ is a rectangle, we say that $\Gamma^s \subset \Gamma$ is an *s-subset* of Γ if $x \in \Gamma^s$ implies $V_x^s \cap \Gamma \subset \Gamma^s$ and $\Gamma^u \subset \Gamma$ is a *u-subset* of Γ if $x \in \Gamma^u$ implies $V_x^u \cap \Gamma \subset \Gamma^u$.

Definition 2.3 (T-return times). For a nice recurrent set A and $x \in A$, let

$$\tau(x) := \min\{i \in \mathbb{N} : f^i(x) \in A\} \quad \text{and} \quad \mathbb{N}_T := \{\tau(x) : x \in A\}$$

⁸As we will see in the next section, for surfaces Young’s tower conditions from [62] turn out to be necessary as well as sufficient, but this was not proved in that paper.

be the *first T-return times* to A .

Definition 2.4 (Topological Young Tower). A nice recurrent rectangle Γ supports a *First T-Return Topological Young Tower* if for each $i \in \mathbb{N}_T$ we can subdivide $\Gamma_i^s := \{x \in \Gamma : \tau(x) = i\}$ into pairwise disjoint s -subsets

$$(Y0) \quad \Gamma_{i1}^s, \dots, \Gamma_{i\kappa_i}^s \quad \text{such that} \quad \Gamma_{ij}^u := f^i(\Gamma_{ij}^s)$$

is a u -subset of Γ for every $j = 1, \dots, \kappa_i$.

Remark 2.5. We remark that the ‘‘First T-Return’’ part of the definition comes from the fact that $\tau(x)$ is the first T-return time. A similar definition can be used for an arbitrary return time to give a more general Topological Young Tower.

Remark 2.6. As mentioned in Remark 1.20 above, several constants, including T , are implicit in the definition of a nice recurrent set. We will sometimes include it in the associated terminology, for example in the notions of ‘‘T-return times’’ and of ‘‘First T-Return’’ in Definitions 2.3 and 2.4 above, when it helps maintain clarity.

2.2. Young Towers. We call the above a *Topological Young Tower* because (Y0) captures only the topological structure of a Young tower which is often, including in the present setting, the most difficult part of the construction. To state the other properties of a Young tower we need to introduce the induced map to Γ .

Definition 2.7 (Induced map). For a rectangle Γ which supports a First T-Return Topological Young Tower, we define the *induced map*

$$F: \Gamma \rightarrow \Gamma \quad \text{by} \quad F|_{\Gamma_i^s} := f^i$$

and refer to Γ as the *base* of the tower. We also let $\text{Jac}^u F(x) := |\det Df^{\tau(x)}|_{E_x^u}|$ denote the unstable Jacobian of this induced map.

Definition 2.8 (Young Tower). Let Γ be a rectangle which supports a First T-Return Topological Young Tower. We say that Γ supports a *First T-Return Young Tower* if there exist constants $\beta_1, \beta_2 \in (0, 1)$ and $c > 0$ such that

- (Y1) for every $i \in \mathbb{N}_T$, $j \in \{1, \dots, \kappa_i\}$, $x \in \Gamma_{ij}^s$:
- (a) $d(F(x), F(y)) \leq \beta_1 d(x, y)$ for every $y \in V_x^s$
 - (b) $d(x, y) \leq \beta_1 d(F(x), F(y))$ for every $y \in V_x^u \cap \Gamma_{ij}^s$.
- (Y2) for every $x \in \Gamma$ and $n \geq 0$:
- (a) For all $y \in V_x^s$, we have

$$\left| \log \frac{\text{Jac}^u F(F^n(x))}{\text{Jac}^u F(F^n(y))} \right| \leq c\beta_2^n;$$

- (b) For all $y \in V_x^u$ such that $F^k(x), F^k(y) \in \Gamma_{i_k j_k}^s$ for some $i_k \in \mathbb{N}_T$, $j_k \in \{1, \dots, \kappa_{i_k}\}$ for all $0 \leq k \leq n$, we have

$$\left| \log \frac{\text{Jac}^u F(F^{n-k}(x))}{\text{Jac}^u F(F^{n-k}(y))} \right| \leq c\beta_2^k.$$

Definition 2.9 (Integrable return times). A *fat* rectangle Γ which supports a First T-Return Young Tower has *integrable return times* if there is $x \in \Gamma$ such that

$$\int_{V_x^u \cap \Gamma} \tau \, dm_{V_x^u} < \infty.$$

2.3. Existence of Young Towers. Young towers are non-trivial geometric structures and their construction generally requires substantial work. A key part of our argument is to show that they are part of the intrinsic structure of hyperbolic sets.

Theorem B. *Let $f: M \rightarrow M$ be a $C^{1+\alpha}$ surface diffeomorphism. For every $\chi > \lambda > 0$, every $0 < \epsilon < \epsilon_1(f, \chi, \lambda)$, and every $\ell \in \mathbb{N}$ there exists $r > 0$ such that if A is a nice recurrent set, then*

- (1) *A is contained in a nice recurrent rectangle $\Gamma \subset \Gamma_{pq}$ which supports a First T-Return Topological Young Tower;*
- (2) *if A is strongly recurrent and the set Γ of part (1) is fat, then Γ supports a First T-Return Young Tower with integrable return times.*

Remark 2.10. A subtle point about the statement of part (2) of Theorem B is that Γ may be fat even though A is not (whereas if A is fat then Γ is also automatically fat by part (1)), and the fatness of Γ is what is needed to construct the tower with integrable return times and thus to imply the existence of an SRB measure. In particular the assumptions in part (1) of Theorem A can be relaxed as we do not need A itself to be fat but just that it generates a fat rectangle Γ through the construction which we will use in the proof of part (1) of Theorem B. A simple setting in which this distinction may be seen to be potentially relevant is that of a two-dimensional Anosov diffeomorphism. Then we can easily find a small nice domain Γ_{pq} and may choose A as a countable dense set of points in Γ_{pq} which means in particular that A is not fat, but we shall see from the proof of Theorem B that this gives rise to a fat rectangle Γ .

Remark 2.11. We stress that the rectangle Γ is nice but not “as nice” as the given set A : A is (χ, ϵ, ℓ) -regular by assumption, whereas Γ is $(\chi', \epsilon', \ell')$ -regular for some $\chi' < \chi$, $\epsilon' > \epsilon$, and $\ell' > \ell$; see Proposition 11.3.

2.4. Historical background. The Young tower approach for constructing invariant measures is a particular case of a general and classical method in ergodic theory, that of *inducing*. This is based on the construction of a return map to some subset of the phase space which is simpler to study and more amenable to the construction of an invariant measure; this invariant measure can then be extended to the whole phase space by an elementary argument.

The specific inducing structure defined above, which is pertinent to the study of systems with hyperbolic behavior, was introduced by Young [62] as a framework for studying the existence and ergodic properties of SRB measures.⁹

⁹A more general inducing structure was introduced in [49]; it can be used to study the existence and ergodic properties of equilibrium measures, which include SRB measures.

Since then, it has been applied to a variety of cases, including billiards [11, 21, 22, 43, 62], certain Hénon maps [15], and partially hyperbolic diffeomorphisms [5, 7, 9, 10]. A non-invertible version of a Young tower was also introduced by Young in [63] and has proved extremely powerful in studying non-invertible maps satisfying nonuniform expansivity conditions [6, 8, 18, 28, 31, 33, 38, 52].

A quite remarkable feature of the method of Young towers is that it associates, by construction, a non-trivial geometric structure to the measure; a structure which moreover can be used effectively to study several statistical and ergodic properties of the measure, see [2, 3, 23, 30, 34, 35, 39, 44, 45, 46, 54, 62, 63] as well as the other references mentioned above. This leads naturally to the question of the domain of applicability of this method: if it implies so much structure then maybe it can only be applied to a limited number of special cases which have this structure? This legitimate doubt is partly supported by the observation that all the constructions of Young towers so far have relied heavily on specific, assumed a-priori, geometric properties of the systems under consideration. This question has therefore led to the so-called *liftability* problem, which is the question of which measures have, or “lift to”, a Young tower structure, see Definition 3.1 below, and therefore can in principle be obtained by the method of Young towers.

For the non-invertible non-uniformly expanding case this has been addressed in several papers and in particular it is shown in [4] that essentially every invariant probability measure which is absolutely continuous with respect to Lebesgue lifts to a Young tower. One consequence of our results, stated formally as Corollary C.1 below, is that for surface diffeomorphisms, every hyperbolic measure (in particular, every SRB measure) lifts to a Young tower, which means that the geometric structure of Young towers is intrinsic to all hyperbolic (and in particular, all SRB) measures. Moreover, we show that every hyperbolic measure lifts to a *first return* Young tower for some iterate of the map, which is perhaps surprising because in most other applications of Young’s work the towers do not have the first return property. Using this, we obtain a way of controlling the tail of the tower, see Corollary C.2.

A related result in this direction is Sarig’s result [56] that every hyperbolic measure for a surface diffeomorphism can be lifted to a countable-state topological Markov shift, which is closely related to a Young tower. The results of [56] have been extended to higher dimensions [47], to three-dimensional flows [42] and to non-uniformly hyperbolic surface maps with discontinuities [41]. It seems natural to expect that our results admit similar extensions, although for the extension to higher dimensions one must confront a major technical obstruction, which is that the nice domains we use are inherently two-dimensional.

3. LIFTING HYPERBOLIC MEASURES TO TOWERS: THEOREM C

Statement (1) of Theorem A follows immediately from Statement (2) of Theorem B and the results in [62] on the existence of SRB measures for Young towers. Statement (2) of Theorem A follows immediately from our next result,

which states that nice strongly recurrent (fat) rectangles are *necessary* for the existence of hyperbolic (SRB) measures.

Theorem C. *Let f be a $C^{1+\alpha}$ surface diffeomorphism. If μ is a non-atomic, hyperbolic (respectively, SRB) measure for f , then for every sufficiently small $\chi > 0$ and every $\epsilon > 0$, there is an integer $\ell \in \mathbb{N}$ such that for every $r > 0$, there exists a nice strongly recurrent (respectively, fat) set A .*

The rest of the paper is therefore devoted to the proofs of Theorems B and C, which we will carry out through the development of some non-trivial technical results of independent interest. We first conclude this section with two almost immediate corollaries of Theorems B and C, also of independent interest. The first is an observation concerning the geometric structure of hyperbolic measures, formalized in the notion of *liftability* which is relevant in many applications and studies of the ergodic properties of invariant measures, see e.g. [50].

Definition 3.1 (Liftable measures). An invariant probability measure μ is *liftable* (to a Topological Young Tower) if there exists a recurrent rectangle Γ which supports a Topological Young Tower and a probability measure $\hat{\mu}$ on Γ which is invariant for the corresponding induced map $F : \Gamma \rightarrow \Gamma$, such that

$$(3.1) \quad E_{\hat{\mu}} := \int_{\Gamma} \tau d\hat{\mu} < \infty \quad \text{and} \quad \mu = \frac{1}{E_{\hat{\mu}}} \sum_{i=0}^{\infty} f_*^i(\hat{\mu}|_{\{\tau > i\}}).$$

The following is an immediate consequence of Theorem C and [64].

Corollary C.1. *Let f be a $C^{1+\alpha}$ surface diffeomorphism. Then every ergodic non-atomic hyperbolic (invariant) probability measure lifts to a First T-Return Topological Young Tower for some $T > 0$.*

Before stating our second corollary, we emphasize that we have a first T-return tower which is, to all intents and purposes, essentially as good as if it was a first return time tower. Existence of a first T-return tower in this generality is a rather surprising result, and has important consequences. For example, because every return to $\Gamma_{pq} \cap \Lambda_\ell$ is a return to the base of the tower, we can relate the size of the tail of the tower to the return rate to regular level sets.

Corollary C.2. *Let f be a $C^{1+\alpha}$ surface diffeomorphism, μ an ergodic non-atomic χ -hyperbolic measure, and Λ a χ -hyperbolic set with $\mu(\Lambda) > 0$. Suppose that for every $\epsilon > 0$ there is $\ell \in \mathbb{N}$ such that for every open set $U \subset M$ with $\mu(U \cap \Lambda_\ell) > 0$ and every $T \in \mathbb{N}$, we have*

$$\overline{\lim}_{n \rightarrow \infty} \frac{1}{n} \log \mu \{x \in U \cap \Lambda_\ell : f^{kT}(x) \notin U \cap \Lambda_\ell \text{ for every } 1 \leq k \leq n\} < 0.$$

Then μ lifts to a First T-Return Topological Young Tower for some $T > 0$, whose tail decays exponentially in the sense that

$$\overline{\lim}_{n \rightarrow \infty} \frac{1}{n} \log \hat{\mu} \{x \in \Gamma : \tau(x) > n\} < 0.$$

Proof of Corollary C.2. Corollary C.2 follows from Theorems B and C by putting $A = \Gamma_{pq} \cap \Lambda_\ell$, $U = \text{Int}(\Gamma_{pq})$, and using the fact that the tower has the first T-return property. \square

Remark 3.2. A similar result can easily be formulated for subexponential rates of decay. If μ is a hyperbolic measure whose log Jacobian along unstable leaves is sufficiently regular, then control of the tail of the tower implies results on decay of correlations and other statistical properties; see [62, 63, 44, 29, 49, 57] for upper bounds on correlations, [55, 32, 57] for lower bounds, and [45, 46] for some other statistical properties (this list is far from comprehensive).

The condition on regularity of the log Jacobian is satisfied when μ is an equilibrium measure for a sufficiently regular potential function. We remark that the results in [25, 26, 19] provide techniques for studying equilibrium states for some classes of non-uniformly hyperbolic systems, which suggest a method for establishing the hypothesis of Corollary C.2. (Note that those systems are not surface diffeomorphisms, so the results here do not apply directly.)

4. HYPERBOLIC BRANCHES FROM ALMOST RETURNS: THEOREM D

A remarkable feature of Theorem B is that its hypotheses contain only a minimal amount of structure whereas the conclusions produce a great deal of non-trivial structure. The fundamental ingredient which we will use to build this structure is that of a *hyperbolic branch* which we proceed to define. The main result of this section, Theorem D, is then a statement on the existence of hyperbolic branches under very mild recurrence conditions.

We assume throughout this section that Λ is a (χ, ϵ) -hyperbolic set and Γ_{pq} is a nice domain as in Figure 1.1.

Definition 4.1 (Cones). Given a normed vector space V , a *cone* in V is any subset $K \subset V$ for which there is a decomposition $V = E \oplus F$ and $\omega > 0$ such that

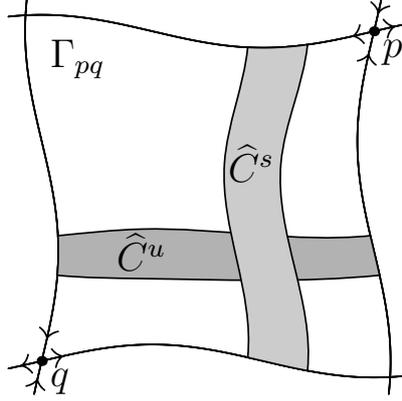
$$K = \{v + w : v \in E, w \in F, \|w\| < \omega\|v\|\}.$$

Say that two cones K and K' are *transverse* if $K \cap K' = \{0\}$. Note that we do not require K and K' to be defined in terms of the same decomposition $E \oplus F$.

Definition 4.2 (Adapted cone fields). A *cone field* over Γ_{pq} is a family of cones $\mathcal{K} = \{K_x \subset T_x M\}_{x \in \Gamma_{pq}}$; it is continuous if the cones K_x can be defined in terms of decompositions $E_x \oplus F_x$ and widths ω_x such that E_x, F_x, ω_x vary continuously in x .

We say that two transverse continuous cone fields $\mathcal{K}^s = \{K_x^s\}$ and $\mathcal{K}^u = \{K_x^u\}$ over Γ_{pq} are *adapted* to the nice set Γ if for every $x \in \Gamma$ the tangent directions at every point of W_x^s belong to \mathcal{K}^s and the tangent directions to every point of W_x^u belong to \mathcal{K}^u .

Definition 4.3 (Stable and unstable strips). Let Γ be a nice set and $\mathcal{K}^s, \mathcal{K}^u$ be adapted cone fields. A C^1 curve $\gamma^s \subset \Gamma_{pq}$ is a *stable curve* if its tangent directions lie in \mathcal{K}^s and a curve $\gamma^u \subset \Gamma_{pq}$ is an *unstable curve* if its tangent


 FIGURE 4.1. Stable strips and unstable strips in Γ_{pq} .

directions lie in \mathcal{K}^u . An stable/unstable curve γ is *full length* if its endpoints lie on the boundary of Γ_{pq} . A region $\widehat{C}^s \subseteq \Gamma_{pq}$ is a *stable strip* if it is bounded by two pieces of W_p^u, W_q^u and two full length stable curves and a region $\widehat{C}^u \subseteq \Gamma_{pq}$ is an *unstable strip* if it is bounded by two pieces of W_p^s, W_q^s and two full length unstable curves; see Figure 4.1

We note that the stable and unstable curves we consider will typically be of the form $W_x^{s/u}$ for some $x \in \Gamma$, but we do not require this as part of the definition.

Definition 4.4 (Hyperbolic Branches). Let $\widehat{C}^s, \widehat{C}^u \subseteq \Gamma_{pq}$ be a stable and an unstable strip respectively and suppose that there exists $i > 0$ such that $f^i(\widehat{C}^s) = \widehat{C}^u$. The map

$$f^i: \widehat{C}^s \rightarrow \widehat{C}^u$$

is a (C, λ) -hyperbolic branch if for every $x \in \widehat{C}^s$ and $y = f^i(x) \in \widehat{C}^u$ we have

$$Df_x^i(K_x^u) \subset K_y^u \quad \text{and} \quad Df_y^{-i}(K_y^s) \subset K_x^s$$

and if for every $v^{s/u} \in \mathcal{K}_x^{s/u}$, the vectors $v_j^{s/u} := Df_x^j(v^{s/u})$ for $0 \leq j \leq i$ satisfy

$$\|v_j^u\| \leq C e^{-\lambda(i-j)} \|v_i^u\| \quad \text{and} \quad \|v_j^s\| \leq C e^{-\lambda j} \|v_0^s\|.$$

As shown in Figure 4.2, hyperbolic branches can easily be “concatenated”, in the following sense. If $f^i: \widehat{C}_1^s \rightarrow \widehat{C}_1^u$ and $f^j: \widehat{C}_2^s \rightarrow \widehat{C}_2^u$ are hyperbolic branches, then $\widehat{C}^s := f^{-i}(\widehat{C}_1^u \cap \widehat{C}_2^s)$ is the stable strip for a hyperbolic branch $f^{i+j}: \widehat{C}^s \rightarrow \widehat{C}^u := f^j(\widehat{C}_1^u \cap \widehat{C}_2^s)$. Thus \widehat{C}^s consists of the points in the stable strip of the first branch that are mapped by f^i to the stable strip of the second branch.

If the two initial branches are both (C, λ) -hyperbolic branches, the concatenation of the two has the property that for every $x \in \widehat{C}^s$ and $v^s \in \mathcal{K}_x^s$, the vectors $v_k^s := Df_x^k(v^s)$ for $0 \leq k \leq i+j$ satisfy $\|v_k^s\| \leq C e^{-\lambda k} \|v_0^s\|$ when $k \leq i$, and

$$\|v_k^s\| \leq C e^{-\lambda(k-i)} \|v_i^s\| \leq C^2 e^{-\lambda k} \|v_0^s\|$$

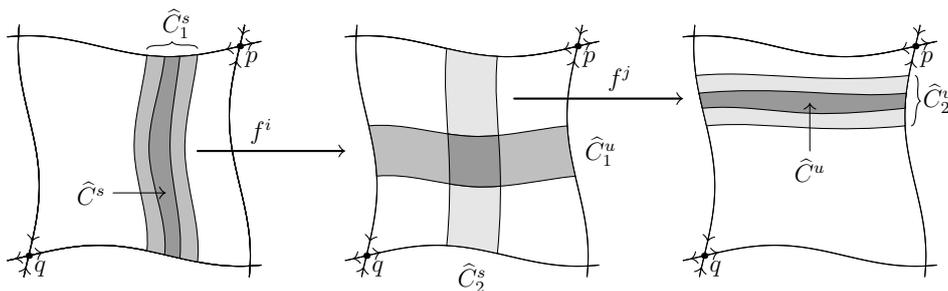


FIGURE 4.2. Concatenating hyperbolic branches.

when $k > i$, with similar estimates on v_k^u ; thus $f^{i+j}: \widehat{C}^s \rightarrow \widehat{C}^u$ is a (C^2, λ) -hyperbolic branch. The fact that C is replaced with C^2 can be a problem as the constant would continue to grow with each repeated concatenation. It turns out, however, that while it is in general impossible to obtain (C, λ) -hyperbolic branches with $C = 1$ (which would avoid any problems with the concatenation), it is often possible to show, because of the way hyperbolic branches are constructed using regular level sets, that the concatenation of any number of (C, λ) -hyperbolic branches is still a (C, λ) -hyperbolic branch.

Definition 4.5 (Concatenation property of hyperbolic branches). We say that a collection of (C, λ) -hyperbolic branches has the *concatenation property* if any finite concatenation of these branches is still a (C, λ) -hyperbolic branch.

The key technical step in the proof of Theorem B is that there exist hyperbolic branches with the concatenation property under some very weak recurrence conditions.

Definition 4.6 (Almost Returns). Consider a nice set Γ . A point $x \in \Gamma$ has an *almost return* to Γ at time $i \in \mathbb{TN}$ (see Figure 4.3) if $f^i(x) \in \Gamma_{pq}$ and there is $y \in \Gamma$ such that

$$[x, y, i] \neq \emptyset \quad \text{where} \quad [x, y, i] := \begin{cases} f^i(W_x^s) \cap W_y^u & \text{if } i > 0 \\ f^{-i}(W_x^u) \cap W_y^s & \text{if } i < 0. \end{cases}$$

We say that Γ is *almost recurrent* if for every $x \in A$ there are $i, j \in \mathbb{TN}$ such that x has an almost return to Γ at times i and $-j$.

Note that actual returns $-x, f^i(x) \in \Gamma$ are special cases of almost returns and thus if Γ is recurrent, then it is also almost recurrent.

Definition 4.7 (Hyperbolic branch property). A nice set Γ has the (C, λ) -*hyperbolic branch property* if there exist adapted cone fields $\mathcal{K}^{s/u}$ such that the following are true:

- (1) whenever $x \in \Gamma$ has an almost return to Γ at a time $i \in \mathbb{TN}$, there exists a (C, λ) -hyperbolic branch $f^i: \widehat{C}^s \rightarrow \widehat{C}^u$ such that $x \in \widehat{C}^s$;
- (2) the collection of such branches has the concatenation property.

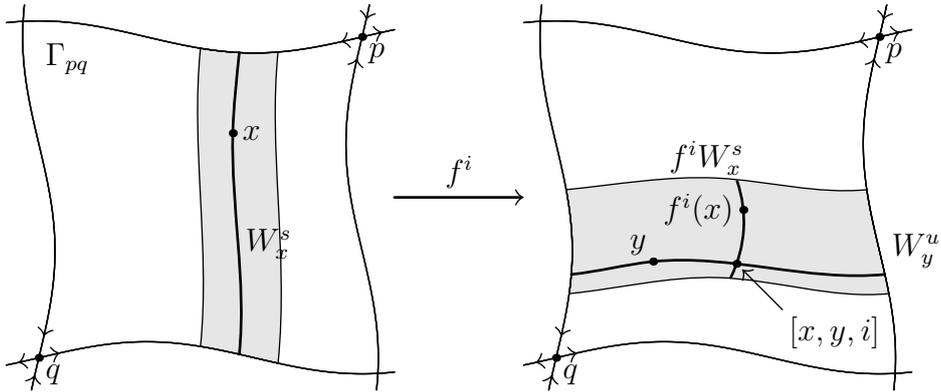


FIGURE 4.3. An almost return and the associated hyperbolic branch.

Theorem D. *Let f be a $C^{1+\alpha}$ surface diffeomorphism. For every $\chi > \lambda > 0$, every $0 < \epsilon < \epsilon_1(f, \chi, \lambda)$, and every $l \in \mathbb{N}$, there is $r > 0$ such that every nice set has the $(\widehat{Q}^{-1}e^{2\epsilon l}, \lambda/3)$ -hyperbolic branch property.*

The constant r determined by Theorem D is the same constant which appears in the statements of Theorem B and Theorem A, and is given relatively explicitly in (10.1). In §11 we prove that Theorem D implies Theorem B, and thus part (1) of Theorem A. The proof does not refer to any of the other results of the paper and can be read independently of the other sections.

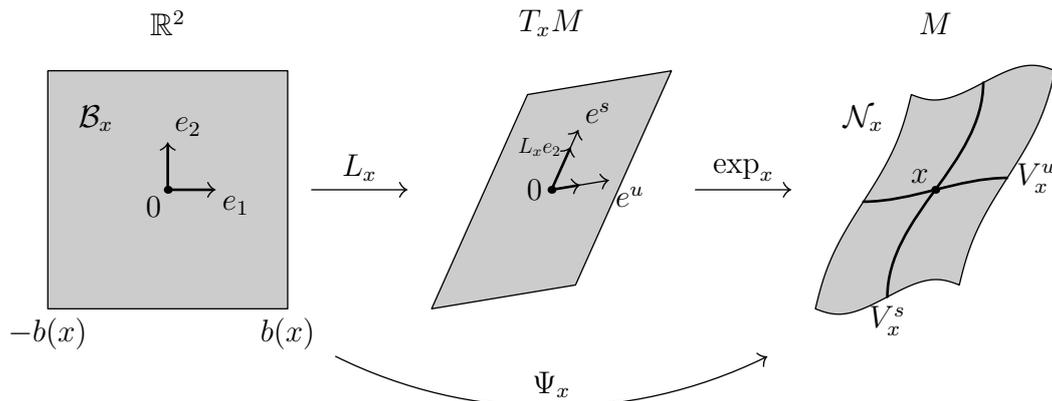
Remark 4.8. The implications Theorem D \Rightarrow Theorem B \Rightarrow Theorem A(1) essentially mean that the existence of a nice strongly recurrent set with the hyperbolic branch property implies the existence of an SRB measure. In the present setting we deduce the hyperbolic branch property from the existence of a global invariant hyperbolic set Λ , but one could envisage verifying this property in particular examples using alternative arguments without necessarily requiring the existence of a global invariant hyperbolic set Λ .

Remark 4.9. The assumption that Γ_{pq} is a nice domain is crucial to prove below that the stable and unstable strips $\widehat{C}^s, \widehat{C}^u$ are contained in Γ_{pq} : this is otherwise not guaranteed, for example if x or $f^i(x)$ are very close to the boundaries of Γ_{pq} .

5. PSEUDO-ORBITS: THEOREMS E AND F

The proofs of Theorems C and D are ultimately based on a new and non-trivial result, Theorem E below, in the general theory of (nonuniformly) hyperbolic sets, which is a generalization of the Katok closing lemma. This result is of independent interest and we expect it to have further applications beyond those presented here. For simplicity, and in view of the setting of our main theorems, we state it for two-dimensional diffeomorphisms but it should generalize in a relatively straightforward way to arbitrary dimension.

In §5.1 we introduce the basic notion of a Lyapunov chart, following the approach of Sarig in [56], and use this to define the notion of stable and

FIGURE 5.1. The Lyapunov chart at x .

unstable strips in §5.2 and of *regular* branch for pseudo-orbits in §5.3. In §5.3 we state our main result about regular branches for pseudo-orbits, and in §5.4 a useful consequence about shadowing orbits.

5.1. Lyapunov charts. Let Λ be a (χ, ϵ) -hyperbolic set. We fix some

$$b > 0$$

sufficiently small to be determined in the course of the proof. This is the only constant which we cannot give *explicitly* in terms of properties of f and the constants associated to the hyperbolic set Λ , however see Remark 5.6, equations (7.13), (7.14), and Lemma 8.14 for the key places in which conditions on b appear. For $x \in \Lambda$, let $e_x^s \in E_x^s$, $e_x^u \in E_x^u$ be unit vectors and define $s, u: \Lambda \rightarrow [1, \infty)$ by

$$(5.1) \quad \begin{aligned} s(x)^2 &:= 2 \sum_{n=0}^{\infty} e^{2n\lambda} \|Df_x^n e_x^s\|^2, \\ u(x)^2 &:= 2 \sum_{n=0}^{\infty} e^{2n\lambda} \|Df_x^{-n} e_x^u\|^2. \end{aligned}$$

By **(H3)** the sums above converge and therefore $s(x), u(x)$ are well defined, though note that they are not uniformly bounded in x . Letting $e_1 = (1, 0)$, $e_2 = (0, 1)$ denote the standard basis vectors in \mathbb{R}^2 , define the linear map $L_x: \mathbb{R}^2 \rightarrow T_x M$, by letting

$$(5.2) \quad L_x(e_1) := u(x)^{-1} e_x^u \quad \text{and} \quad L_x(e_2) := s(x)^{-1} e_x^s$$

and extending to \mathbb{R}^2 by linearity. We call L_x the *Lyapunov change of coordinates* at x . In Lemma 6.4 we prove the following standard relation between the Riemannian metric and the metric induced by the Lyapunov coordinates:

for any $0 < \lambda < \chi$ let¹⁰

$$(5.3) \quad Q_0 := 1/8 \quad \text{and} \quad \widehat{Q} := Q_0 \left(2 \sum_{i=0}^{\infty} e^{2(\lambda-\chi)i} \right)^{-1/2},$$

then, for every $\ell \in \mathbb{N}$ and $x \in \Lambda_\ell$, we have

$$(5.4) \quad 1 \leq \|L_x^{-1}\| \leq 3Q_0\widehat{Q}^{-1}e^{2\epsilon\ell}.$$

For every $x \in \Lambda$ let

$$(5.5) \quad b(x) := b \left(\sum_{k=-\infty}^{\infty} e^{-3|k|\epsilon} \|L_{f^k(x)}^{-1}\| \right)^{-\frac{1}{\alpha}}.$$

Notice that the sum converges by (1.3) and (5.4). For $\ell \geq 1$ let

$$(5.6) \quad b_\ell := b \left(3Q_0\widehat{Q}^{-1}e^{2\epsilon\ell} \sum_{k=-\infty}^{\infty} e^{-|k|\epsilon} \right)^{-\frac{1}{\alpha}}.$$

It follows immediately from the definition that $b_{\ell+1}/b_\ell = e^{-2\epsilon/\alpha}$ and $b \geq b(x) \geq b_\ell > 0$, and it follows from (5.4) (we give a formal proof in Lemma 6.5) that $e^{-3\epsilon/\alpha} < b(x)/b(f(x)) < e^{3\epsilon/\alpha}$. Then, for every $x \in \Lambda_\ell$ we define

$$(5.7) \quad \mathcal{B}_x^{(\ell)} := [-b_\ell, b_\ell]^2 \subseteq [-b(x), b(x)]^2 =: \mathcal{B}_x \subset \mathbb{R}^2.$$

Letting $\exp_x: T_x M \rightarrow M$ be the exponential map, define $\Psi_x: \mathcal{B}_x \rightarrow M$ by

$$\Psi_x := \exp_x \circ L_x.$$

The map Ψ_x is called a *Lyapunov chart* at x ; see [13] for a more general notion.¹¹ We write

$$(5.8) \quad \mathcal{N}_x^{(\ell)} := \Psi_x(\mathcal{B}_x^{(\ell)}) \subseteq \Psi_x(\mathcal{B}_x) =: \mathcal{N}_x.$$

Notice that $\Psi_x(0) = x$ and therefore $\mathcal{N}_x^{(\ell)}, \mathcal{N}_x$ are neighbourhoods of x , which we call respectively the *regular neighbourhood of level ℓ* of x and the *regular neighbourhood* of x .

5.2. Stable and unstable strips. We want to define stable and unstable cones and other objects related to these regular neighbourhoods and Lyapunov charts. For this we need to introduce an additional small constant ω which, for completeness, we define precisely.

Let $\chi > \lambda > 0$ and $0 < \epsilon < \epsilon_1(f, \chi, \lambda)$, and Λ a (χ, ϵ) -hyperbolic set. Notice that (1.7) gives $\epsilon < \alpha\lambda/18$ and so $3\epsilon/\alpha < \lambda/6$, which gives $e^{-\lambda/2}e^{3\epsilon/\alpha} < e^{-\lambda/3}$.

¹⁰We could of course replace Q_0 in the expression for \widehat{Q} by its explicit value but various calculations to be given below will be easier and clearer by keeping track of Q_0 as an independent constant.

¹¹In [13] the Lyapunov change of coordinates L_x is required to be *tempered*, but we do not require this condition.

Thus we can choose $\omega > 0$ sufficiently small that the following inequalities all hold:

$$(5.9) \quad \begin{aligned} e^{-\lambda/2}e^{3\epsilon/\alpha} + e^{-\lambda}\omega &< e^{-\lambda/3}, & (1 - \omega) &\geq \sqrt{2(1 + \omega^2)}/2, \\ e^\lambda/\sqrt{1 + \omega^2} &\geq e^{2\lambda/3}, & 2\omega &< 1 - e^{\lambda/24}e^{-\lambda/3}, \\ (e^{-\lambda/24} - \omega e^{\lambda/24})/\sqrt{1 + \omega^2} &> e^{-\lambda/4}, & e^{-\lambda/24} &< 1/\sqrt{1 + e^{-2\lambda}\omega^2}. \end{aligned}$$

From now on we always assume that ω satisfies (5.9).

Definition 5.1 (Stable and unstable cones). For any $\lambda > 0$ we fix the following cones defined in terms of standard Euclidean coordinates with $v = (v_1, v_2) \in \mathbb{R}^2$:

$$(5.10) \quad \begin{aligned} \tilde{K}^s &:= \{v : |v_1| < e^{-\lambda}\omega|v_2|\} \subset \{v : |v_1| < \omega|v_2|\} := K^s, \\ \tilde{K}^u &:= \{v : |v_2| < e^{-\lambda}\omega|v_1|\} \subset \{v : |v_2| < \omega|v_1|\} := K^u. \end{aligned}$$

Given $x \in \Lambda$ and $y \in \mathcal{N}_x$, we write

$$(5.11) \quad K_{x,y}^{s/u} := D_{\Psi_x^{-1}(y)}\Psi_x(K^{s/u}) \quad \text{and} \quad \tilde{K}_{x,y}^{s/u} := D_{\Psi_x^{-1}(y)}\Psi_x(\tilde{K}^{s/u})$$

for the cones in T_yM that correspond to the cones over \mathcal{B}_x .

Definition 5.2 (Stable and unstable curves). Let $x \in \Lambda$. A C^1 curve $\gamma^s \subset \mathcal{B}_x$ is a *stable curve* (resp. *strongly stable curve*) if its tangent directions lie in K^s (resp. \tilde{K}^s); similarly for unstable curves $\gamma^u \subset \mathcal{B}_x$. In particular, the horizontal and vertical boundaries of \mathcal{B}_x or $\mathcal{B}_x^{(\ell)}$, are stable and unstable curves respectively and so we denote them by $\partial^s\mathcal{B}_x, \partial^s\mathcal{B}_x^{(\ell)}$ and $\partial^u\mathcal{B}_x, \partial^u\mathcal{B}_x^{(\ell)}$ respectively. A stable curve γ^s is a *full length stable curve*, with respect to \mathcal{B}_x or $\mathcal{B}_x^{(\ell)}$ if its endpoints lie on distinct components of $\partial^u\mathcal{B}_x$ or $\partial^u\mathcal{B}_x^{(\ell)}$ respectively. We define *full length unstable curves* similarly. For a C^1 curve $\gamma^{s/u} \subset \mathcal{N}_x$ we use the same terminology according to the geometry of the corresponding curve $\Psi_x^{-1}(\gamma^{s/u}) \subset \mathcal{B}_x$.

Definition 5.3 (Stable and unstable strips). Let $x \in \Lambda$. A region $\mathcal{B}_x^s \subseteq \mathcal{B}_x^{(\ell)}$ is a (*strongly*) *stable strip* if its boundary is formed by two full length (strongly) stable curves and two pieces of $\partial^u\mathcal{B}_x^{(\ell)}$. Full length unstable curves and (*strongly*) *unstable strips* $\mathcal{B}_x^u \subseteq \mathcal{B}_x^{(\ell)}$ are defined similarly. Moreover, for subsets $\mathcal{N}_x^{s/u} \subseteq \mathcal{N}_x^{(\ell)}$, we use the same terminology depending on the geometry of the corresponding sets $\Psi_x^{-1}(\mathcal{N}_x^{s/u}) \subseteq \mathcal{B}_x^{(\ell)}$.

5.3. Pseudo-orbits and regular branches.

Definition 5.4 (Pseudo-orbits). Given constants $\delta, \lambda > 0$ and a finite sequence $\bar{\ell} = (\ell_0, \dots, \ell_k)$ of positive integers satisfying $|\ell_j - \ell_{j-1}| \leq 1$ for all $1 \leq j \leq k$, we say that a finite sequence of points $\bar{x} = (x_0, \dots, x_k)$ is an $(\bar{\ell}, \delta, \lambda)$ -*pseudo-orbit* if for every $0 \leq j \leq k$ we have

$$x_j \in \Lambda_{\ell_j} \quad \text{and} \quad d(f(x_{j-1}), x_j) \leq \delta e^{-\lambda\ell_j}.$$

Notice that the true orbit of the point $x \in \Lambda_\ell$ is trivially an $(\bar{\ell}, \delta, \lambda)$ -pseudo-orbit for $\delta = 0$ and for the sequence $\ell_j = \ell + j$, recall (1.3). The results we present here are essentially already known for true orbits, but their generalizations to pseudo-orbits is non-trivial and constitute a crucial step in our arguments. Given an $(\bar{\ell}, \delta, \lambda)$ -pseudo-orbit we write

$$\mathcal{N}_{\bar{x}}^0 := \bigcap_{i=0}^k f^{-i} \mathcal{N}_{x_i}^{(\ell_i)} \quad \text{and} \quad \mathcal{N}_{\bar{x}}^j := f^j(\mathcal{N}_{\bar{x}}^0) \text{ for } 1 \leq j \leq k$$

for the sets of points corresponding to orbit segments that go through all the Lyapunov neighborhoods of the points x_i ; the size of these sets depends on the choice of $\bar{\ell}$, but we suppress this in the notation. In Lyapunov coordinates, for every $0 \leq j \leq k$, we write

$$(5.12) \quad \mathcal{B}_{\bar{x}}^j := \Psi_{x_j}^{-1}(\mathcal{N}_{\bar{x}}^j) \subseteq \mathcal{B}_{x_j}^{(\ell_j)}$$

and for every $0 \leq i, j \leq k$,

$$(5.13) \quad f_{\bar{x}}^{i,j} := \Psi_{x_j}^{-1} \circ f^{j-i} \circ \Psi_{x_i} : \mathcal{B}_{\bar{x}}^i \rightarrow \mathcal{B}_{\bar{x}}^j,$$

which is a diffeomorphism between $\mathcal{B}_{\bar{x}}^i$ and $\mathcal{B}_{\bar{x}}^j$ by definition. We will show that such maps are hyperbolic branches in a suitable sense.

Definition 5.5 ($\bar{\ell}$ -regular branch). An $(\bar{\ell}, \delta, \lambda)$ -pseudo-orbit \bar{x} determines an $\bar{\ell}$ -regular branch for f if the following are true:

- (i) $\mathcal{B}_{\bar{x}}^0, \mathcal{B}_{\bar{x}}^k$ are stable and unstable strips in $\mathcal{B}_{x_0}^{(\ell_0)}, \mathcal{B}_{x_k}^{(\ell_k)}$, respectively;
- (ii) given $0 \leq i < j \leq k$, $\underline{y} \in \mathcal{B}_{\bar{x}}^i, \underline{z} \in \mathcal{B}_{\bar{x}}^j, \underline{v}^u \in K^u$, and $\underline{v}^s \in K^s$, we have

$$(5.14) \quad \begin{aligned} D_{\underline{y}} f_{\bar{x}}^{i,j} \underline{v}^u &\in K^u, & \|D_{\underline{y}} f_{\bar{x}}^{i,j} \underline{v}^u\| &\geq e^{\lambda(j-i)/3} \|\underline{v}^u\|, \\ D_{\underline{z}} f_{\bar{x}}^{j,i} \underline{v}^s &\in K^s, & \|D_{\underline{z}} f_{\bar{x}}^{j,i} \underline{v}^s\| &\geq e^{\lambda(j-i)/3} \|\underline{v}^s\|. \end{aligned}$$

In the specific case where $\ell_0 = \ell_k = \ell$ and $\ell_j = \min(\ell + j, \ell + k - j)$ for $0 < j < k$, we refer to this as an ℓ -regular branch.

Remark 5.6. The hyperbolicity estimates in (5.14) are given at the level of the Lyapunov charts, and not on the surface itself; the relationship between the two is given by the following simple estimate. Since $D_0 \exp_x$ is the identity map, we can choose b small enough that $\Psi_x : \mathcal{B}_x \rightarrow \mathcal{N}_x$ is a diffeomorphism, in particular injective, for every $x \in \Lambda$, and such that (5.4) gives

$$(5.15) \quad \|D_{\underline{y}} \Psi_x\| \leq 2 \quad \text{and} \quad \|D_{\underline{y}} \Psi_x^{-1}\| \leq 4Q_0 \widehat{Q}^{-1} e^{2\epsilon\ell}$$

for every $\ell \in \mathbb{N}$, $x \in \Lambda_\ell$, $\underline{y} \in \mathcal{B}_x$, and $y \in \mathcal{N}_x$. Recalling that $Q_0 = 1/8$, from (5.3), this implies that if $\bar{x} = (x_0, \dots, x_k)$ determines an $\bar{\ell}$ -regular branch, then given any $y \in \mathcal{N}_{\bar{x}}^0$, $0 < i \leq k$, $z = f^i(y) \in \mathcal{N}_{\bar{x}}^i$, $v^u \in K_{x_0, y}^u$, and $v^s \in K_{x_i, z}^s$,

$$(5.16) \quad \begin{aligned} D_y f^i v^u &\in K_{x_i, z}^u, & \|D_y f^i v^u\| &\geq \widehat{Q} e^{-2\epsilon\ell} e^{\lambda i/3} \|v^u\|, \\ D_y f^{-i} v^s &\in K_{x_0, y}^s, & \|D_y f^{-i} v^s\| &\geq \widehat{Q} e^{-2\epsilon\ell} e^{\lambda i/3} \|v^s\|. \end{aligned}$$

In particular notice that the multiplicative constant depends only on the level ℓ of the *initial* point of the pseudo-orbit. The advantage of writing the hyperbolicity estimates in Lyapunov coordinates is that they can be concatenated

any number of times with no loss of hyperbolicity, which is not the case for the estimates (5.16) on the surface due to the multiplicative constant. This will be crucial in our applications of regular branches to the proof of Theorem D.

We are now ready to state our main result in the general setting of hyperbolic sets.

Theorem E. *Let $f: M \rightarrow M$ be a $C^{1+\alpha}$ surface diffeomorphism. For every $\chi > \lambda > 0$ and every $0 < \epsilon < \epsilon_1(f, \chi, \lambda)$, there are $b, \delta > 0$ such that if Λ is a (χ, ϵ) -hyperbolic set then every $(\bar{\ell}, \delta, \lambda)$ -pseudo-orbit in Λ determines an $\bar{\ell}$ -regular branch.*

In the theory of *uniformly* hyperbolic systems, it is well-known that every pseudo-orbit segment determines a regular branch as in Definition 5.5; that is, there is $\delta > 0$ such that x_0, \dots, x_k determines a regular branch whenever $d(f(x_j), x_{j+1}) < \delta$ for all $0 \leq j < k$. In the setting of non-uniform hyperbolicity, various versions of Theorem E have been obtained. The first result of this type is the well known Katok closing lemma [40]. Other versions were obtained by Hirayama [36], and by Sarig [56] in his construction of countable Markov partitions for surface diffeomorphisms with positive topological entropy. What makes our Theorem E different is the *explicit* relationship between ℓ and the pseudo-orbit scale $\delta e^{-\lambda \ell}$; this is absolutely crucial for our arguments, in particular, for our construction of hyperbolic branches associated with almost T -returns.

5.4. Shadowing. We conclude this section by stating a consequence of Theorem E for the problem of *shadowing* in the non-uniformly hyperbolic setting. It is classical problem in hyperbolic dynamics whether every pseudo-orbit is “shadowed” by a real orbit of the system. In uniform hyperbolicity theory a positive answer to this question is a fundamental result, with many applications and in particular, is a key ingredient in the construction of Markov partitions. Extending it to the setting of non-uniform hyperbolicity has proved challenging and few results have been obtained, see [13]. Here we give a powerful shadowing result, which follows relatively easily from Theorem E. In addition to the existence of a shadowing orbit, this result gives an explicit estimate on the hyperbolicity constants associated to this shadowing orbit and also provides a “weak” version of Theorem 1.11 on the existence of Lipschitz continuous local stable and unstable curves, see Remark 5.8.

Definition 5.7. Given a bi-infinite sequence $\bar{\ell} = (\ell_n)_{n \in \mathbb{Z}}$ with $|\ell_{n+1} - \ell_n| \leq 1$ for all n , a *bi-infinite $(\bar{\ell}, \delta, \lambda)$ -pseudo-orbit* is a bi-infinite sequence $(x_n)_{n \in \mathbb{Z}}$ such that for every $n \in \mathbb{Z}$, we have $x_n \in \Lambda_{\ell_n}$ and $d(f(x_{n-1}), x_n) \leq \delta e^{-\lambda \ell_n}$.

Theorem F. *Let $f: M \rightarrow M$ be a $C^{1+\alpha}$ surface diffeomorphism. For every $\chi > \lambda > 0$ and every $0 < \epsilon < \epsilon_1(f, \chi, \lambda)$, there are $b, \delta > 0$ such that if Λ is a (χ, ϵ) -hyperbolic set and $(x_n)_{n \in \mathbb{Z}}$ is a bi-infinite $(\bar{\ell}, \delta, \lambda)$ -pseudo-orbit in Λ , then there is a unique shadowing point $y \in M$ such that $f^n(y) \in \Psi_{x_n}(\mathcal{B}_{x_n}^{(\ell_n)})$ for all $n \in \mathbb{Z}$. Moreover, the point y is $(\lambda/4, 2\epsilon, \ell_0 + \ell')$ -regular for $\ell' = \lceil \frac{1}{2\epsilon} \log \widehat{Q} \rceil$.*

Let σ denote the left shift on $\mathbb{N}^{\mathbb{Z}}$ and $\Lambda^{\mathbb{Z}}$. If \bar{x} is a bi-infinite $(\bar{\ell}, \delta, \lambda)$ -pseudo-orbit, then $\sigma\bar{x}$ is a bi-infinite $(\sigma\bar{\ell}, \delta, \lambda)$ -pseudo-orbit. It follows from uniqueness of the shadowing point that the map $\bar{x} \mapsto y$ intertwines σ and f : if y is the unique shadowing point for \bar{x} , then $f(y)$ is the unique shadowing point for $\sigma\bar{x}$. Moreover, if the pseudo-orbit is periodic in the sense that $\sigma^n\bar{\ell} = \bar{\ell}$ and $\sigma^n\bar{x} = \bar{x}$ for some $n \in \mathbb{N}$, then $f^n(y) = y$.

Remark 5.8. The set of points that shadow the forward infinite pseudo-orbit x_0, x_1, x_2, \dots is the local stable curve V_y^s of y , and similarly for the backward pseudo-orbit and the local unstable curve. Observe that $V_y^s = \bigcap_{n \geq 0} \mathcal{N}_{(x_0, \dots, x_n)}^0$, where $\mathcal{N}_{(x_0, \dots, x_n)}^0$ is the stable strip associated to the $\bar{\ell}$ -regular branch guaranteed by applying Theorem E to the pseudo-orbit x_0, \dots, x_n . The boundaries of these stable strips can be represented in the coordinates \mathcal{B}_{x_0} as graphs of ω -Lipschitz functions from the span of e_2 to the span of e_1 (see Figures 5.1 and 7.1). It is easy to show that these functions converge to an ω -Lipschitz function whose graph gives V_y^s . This establishes a “weak” version of the Stable Manifold Theorem; for the full version, which gives differentiability of V_y^s , one needs to do some further work to control the regularity properties of the *graph transform*.

Corollary 5.9. *If $x, y \in \Lambda_\ell$ and $d(x, y) \leq \delta e^{-\lambda\ell}$, then writing*

$$\ell_n = \ell + |n| \quad \text{and} \quad z_n = \begin{cases} f^{-n}(y) & n < 0, \\ f^n(x) & n \geq 0 \end{cases}$$

gives a $(\bar{\ell}, \delta, \lambda)$ -pseudo-orbit \bar{z} , whose unique shadowing point z is the bracket $[x, y] = V_x^s \cap V_y^u$. Thus $\{[x, y] : x, y \in \Lambda_\ell, d(x, y) \leq \delta e^{-\lambda\ell}\}$ is a $(\lambda/4, 2\epsilon, \ell + \ell')$ -regular set, where $\ell' = \lceil |\log \widehat{Q}| / \epsilon \rceil$.

Remark 5.10. The δ in Corollary 5.9 is the same as in Theorem F. This corollary in particular shows that the bracket $[x, y]$ exists, so that $\delta e^{-\lambda\ell}$ plays the role of δ_ℓ from Theorem 1.11. In general the δ_ℓ used there could be larger than $\delta e^{-\lambda\ell}$, because that result made no claims about regularity of $[x, y]$.

Part II. Hyperbolic Theory

In this part of the paper we develop all the general hyperbolic theory needed for the proofs of our main results, and prove Theorems E and F. For simplicity we state and prove everything in the two-dimensional setting, as required by our applications of these results, but we expect all the arguments to generalize to arbitrary dimensions. The contents of this part are completely self-contained, with no reference to existing results in the literature, and follow directly from the definition of (χ, ϵ) -hyperbolic set.

In §6 we give some basic estimates related to Lyapunov charts. In §7 we state and prove Theorem G, on the hyperbolicity of f in Lyapunov charts, which is a fundamental result in the theory of hyperbolic sets and the key motivation behind the introduction of Lyapunov charts. In §8 we state and

prove Theorem H which gives some conditions guaranteeing that Lyapunov charts of nearby points are “overlapping” in a suitable sense. Some qualitative versions of this result are known but we give a quantitative version which is not available in the existing literature and which is crucial for our arguments; this is the most involved technical step in the paper. In §9.1 we combine these two results to prove Theorem E, and in §9.2 we deduce Theorem F.

Throughout the proofs, we will write Q_j for various constants that depend only on $f, \chi, \lambda, \epsilon$ and are independent of x, y, ℓ, n, k , etc.

6. LYAPUNOV CHART ESTIMATES

Recall the definition of Lyapunov charts in §5.1 and in particular the functions $s(x), u(x)$ defined in (5.1), the map L_x defined in (5.2), and the quantities b_ℓ and $b(x)$ defined in (5.6) and (5.5) respectively. We follow here the basic approach of Sarig in [56] and prove some properties of these objects, including (5.4). We start with a couple of simple estimates which show that although the functions $s(x), u(x)$ are not slowly varying along orbits, they are uniformly bounded on each Λ_ℓ and satisfy a bounded variation property along orbits.

Lemma 6.1. *For every $\ell \geq 1$ and $x \in \Lambda_\ell$, we have*

$$(6.1) \quad \sqrt{2} \leq s(x) \leq Q_0 \widehat{Q}^{-1} e^{\epsilon \ell} \quad \text{and} \quad \sqrt{2} \leq u(x) \leq Q_0 \widehat{Q}^{-1} e^{\epsilon \ell}.$$

Proof. The lower bounds follow immediately from the definition of $s(x), u(x)$ in (5.1). Using the hyperbolicity property **(H3)** and the definition of Q_0, \widehat{Q} in (5.3), we have

$$s(x)^2 \leq 2 \sum_{n \geq 0} e^{2n\lambda} C(x)^2 e^{-2n\chi} = C(x)^2 (Q_0 \widehat{Q}^{-1})^2,$$

and then the definition of Λ_ℓ in (1.2) gives the upper bound for $s(x)$. The upper bound for $u(x)$ is similar. \square

Lemma 6.2. *There exists a constant $Q_1 = Q_1(c_1, c_2, \lambda) > 0$ such that for every $x \in \Lambda$ and unit vectors $e_x^s \in E_x^s, e_x^u \in E_x^u$, we have*

$$\begin{aligned} Q_1^{-1} &\leq e^\lambda \|D_x f(e_x^s)\| \leq s(x)/s(f(x)) \leq \sqrt{1 + e^{2\lambda} \|D_x f(e_x^s)\|^2} \leq Q_1, \\ Q_1^{-1} &\leq e^\lambda \|D_x f(e_x^u)\|^{-1} \leq u(f(x))/u(x) \leq \sqrt{1 + e^{2\lambda} \|D_x f(e_x^u)\|^{-2}} \leq Q_1. \end{aligned}$$

Proof. For $s(x)$ we have

$$\frac{s(x)^2}{2} := \sum_{k=0}^{\infty} e^{2k\lambda} \|D f_x^k e_x^s\|^2 = 1 + e^{2\lambda} \|D f_x e_x^s\|^2 \sum_{k=1}^{\infty} e^{2(k-1)\lambda} \|D f_{f(x)}^{k-1} e_{f(x)}^s\|^2,$$

and

$$\sum_{k=1}^{\infty} e^{2(k-1)\lambda} \|D f_{f(x)}^{k-1} e_{f(x)}^s\|^2 = \sum_{k=0}^{\infty} e^{2k\lambda} \|D f_{f(x)}^k e_{f(x)}^s\|^2 = s(f(x))^2,$$

which gives $s(x)^2 = 2(1 + e^{2\lambda}\|Df_x e_x^s\|^2 s(f(x))^2)$. Rearranging gives

$$(6.2) \quad \frac{s(x)}{s(f(x))} = \sqrt{2(s(f(x)))^{-2} + e^{2\lambda}\|Df_x e_x^s\|^2} \geq e^\lambda \|Df_x e_x^s\|,$$

and the upper bound for $s(x)/s(f(x))$ also follows since $s(f(x)) \geq \sqrt{2}$ so $2(s(f(x)))^{-2} \leq 1$. For uniformity of Q_1 it suffices to note that $\|Df_x^{\pm 1}\| \leq \max(e^{-c_1}, e^{c_2})$ by (1.5). A similar computation for $u(x)$, using f^{-1} in place of f , gives the following analogue of (6.2):

$$\frac{u(x)}{u(f^{-1}(x))} = \sqrt{2(u(f^{-1}(x)))^{-2} + e^{2\lambda}\|Df_x^{-1} e_x^u\|^2} \geq e^\lambda \|Df_x^{-1} e_x^u\|,$$

with the upper bound again coming from $u(f(x)) \geq \sqrt{2}$, so that

$$e^\lambda \|Df_x^{-1} e_x^u\| \leq \frac{u(x)}{u(f^{-1}(x))} \leq \sqrt{1 + e^{2\lambda}\|Df_x^{-1} e_x^u\|^2}.$$

Applying this to $f(x)$ and $f^{-1}(f(x)) = x$, we get

$$e^\lambda \|Df_{f(x)}^{-1} e_{f(x)}^u\| \leq \frac{u(f(x))}{u(x)} \leq \sqrt{1 + e^{2\lambda}\|Df_{f(x)}^{-1} e_{f(x)}^u\|^2}.$$

Using that $\|Df_{f(x)}^{-1} e_{f(x)}^u\| = \|Df_x e_x^u\|^{-1}$ we get the bounds for u , and uniformity via Q_1 comes as it did for s . \square

An almost immediate, but extremely important, consequence of Lemma 6.2, and therefore of the way the functions $s(x), u(x)$ are defined, is the following fundamental result originally proved in [48].¹²

Theorem 6.3 (Oseledets–Pesin Reduction Theorem). *For every $x \in \Lambda$,*

$$(6.3) \quad L_{f(x)}^{-1} \circ D_x f \circ L_x = \begin{pmatrix} A_x & 0 \\ 0 & B_x \end{pmatrix},$$

where L_x is given by (5.2) and $A_x, B_x \in \mathbb{R}$ satisfy

$$(6.4) \quad 0 < Q_1^{-1} < B_x \leq e^{-\lambda} < 1 < e^\lambda \leq A_x < Q_1$$

Proof. The diagonal form is a consequence of the invariance of the stable and unstable subspaces E_x^s, E_x^u and the fact that $L_x, L_{f(x)}$ map the coordinate axes to these subspaces, recall (5.2). Thus, by linearity of L_x we have

$$\begin{aligned} A_x e_1 &= L_{f(x)}^{-1} \circ D_x f \circ L_x(e_1) = L_{f(x)}^{-1} \circ D_x f \left(\frac{e_x^u}{u(x)} \right) = \frac{u(f(x))}{u(x)} \|D_x f(e_x^u)\|, \\ B_x e_2 &= L_{f(x)}^{-1} \circ D_x f \circ L_x(e_2) = L_{f(x)}^{-1} \circ D_x f \left(\frac{e_x^s}{s(x)} \right) = \frac{s(f(x))}{s(x)} \|D_x f(e_x^s)\|, \end{aligned}$$

and the statement then follows from Lemma 6.2. \square

¹²In [48] it is required that L_x is tempered, but this is not necessary for our formulation.

Lemma 6.4. *For every $x \in \Lambda$, we have*

$$(6.5) \quad 1 \leq \frac{\sqrt{s(x)^2 + u(x)^2}}{\sqrt{2} \sin \angle(E_x^s, E_x^u)} \leq \|L_x^{-1}\| \leq \frac{\sqrt{s(x)^2 + u(x)^2}}{\sin \angle(E_x^s, E_x^u)}.$$

In particular, for every $\ell \geq 1$, $x \in \Lambda_\ell$, and $k \in \mathbb{Z}$, we have

$$(6.6) \quad 1 \leq \|L_{f^k(x)}^{-1}\| \leq 3Q_0\widehat{Q}^{-1}e^{2\epsilon\ell}e^{2\epsilon|k|}.$$

Proof. Let $\theta(x) = \angle(E_x^s, E_x^u)$. Consider the orthonormal basis $\{e_x^u, (e_x^u)^\perp\}$ in T_xM , oriented so that $e_x^s = \cos \theta(x)e_x^u + \sin \theta(x)(e_x^u)^\perp$. From (5.2), we have $L_x e_1 = u(x)^{-1}e_x^u$ and $L_x e_2 = s(x)^{-1}e_x^s$, so the matrices of $L_x^{\pm 1}$ relative to the orthonormal bases $\{e_1, e_2\}$ and $\{e_x^u, (e_x^u)^\perp\}$ have the form

$$L_x = \begin{pmatrix} u(x)^{-1} & s(x)^{-1} \cos \theta(x) \\ 0 & s(x)^{-1} \sin \theta(x) \end{pmatrix} \quad \text{and} \quad L_x^{-1} = \begin{pmatrix} u(x) & -u(x)/\tan \theta(x) \\ 0 & s(x)/\sin \theta(x) \end{pmatrix}.$$

The norm of $A = L_x^{-1}$ is the square root of the largest eigenvalue of

$$A^T A = \frac{1}{\sin^2 \theta(x)} \begin{pmatrix} u(x)^2 & -s(x)u(x) \cos \theta(x) \\ -s(x)u(x) \cos \theta(x) & s(x)^2 \end{pmatrix},$$

and thus a routine computation with the quadratic formula gives

$$\|L_x^{-1}\|^2 = \frac{u(x)^2 + s(x)^2 + \sqrt{(u(x)^2 + s(x)^2)^2 - 4s(x)^2u(x)^2 \sin^2 \theta(x)}}{2 \sin^2 \theta(x)}.$$

The square root term lies between 0 and $u(x)^2 + s(x)^2$, which proves (6.5).

For (6.6), we first observe that $\sin \theta \geq \frac{2\theta}{\pi}$ for all $\theta \in [0, \frac{\pi}{2}]$. From (1.3), we see that if $x \in \Lambda_\ell$ then $f^k(x) \in \Lambda_{\ell+|k|}$ and so $\angle(E_{f^k(x)}^s, E_{f^k(x)}^u) \geq e^{-\epsilon(\ell+|k|)}$ and, by (6.1), $s(f^k(x)) \leq Q_0\widehat{Q}^{-1}e^{\epsilon(\ell+|k|)}$ and $u(f^k(x)) \leq Q_0\widehat{Q}^{-1}e^{\epsilon(\ell+|k|)}$. Substituting these bounds into the upper bound from (6.5) gives

$$\|L_{f^k(x)}^{-1}\| \leq \frac{\sqrt{2}Q_0\widehat{Q}^{-1}e^{\epsilon(\ell+|k|)}}{2e^{-\epsilon(\ell+|k|)}/\pi} = \frac{\pi}{\sqrt{2}}Q_0\widehat{Q}^{-1}e^{2\epsilon\ell}e^{2\epsilon|k|},$$

which proves (6.6) since $\pi/\sqrt{2} < 3$. \square

Lemma 6.5. *For every $\ell \geq 1$ and $x \in \Lambda_\ell$*

$$b \geq b(x) \geq b_\ell > 0 \quad \text{and} \quad e^{-3\epsilon/\alpha} < b(x)/b(f(x)) < e^{3\epsilon/\alpha}.$$

Proof. From (6.6), the sum in the definition of $b(x)$ converges. Therefore, for ϵ small and $x \in \Lambda_\ell$ we have

$$1 \leq \|L_x^{-1}\| \leq \sum_{k=-\infty}^{\infty} e^{-3|k|\epsilon} \|L_{f^k(x)}^{-1}\| \leq 3Q_0\widehat{Q}^{-1}e^{2\epsilon\ell} \sum_{k=-\infty}^{\infty} e^{-|k|\epsilon}$$

and thus $b \geq b(x) \geq b_\ell > 0$ as in the first part of the statement. Moreover

$$b(f(x)) := b \left(\sum_{k=-\infty}^{\infty} e^{-3|k|\epsilon} \|L_{f^{k+1}(x)}^{-1}\| \right)^{-1/\alpha}$$

$$\begin{aligned}
&= b \left(\sum_{k=-\infty}^{\infty} e^{-3|k-1|\epsilon} \|L_{f^k(x)}^{-1}\| e^{3(-|k|+|k|)\epsilon} \right)^{-1/\alpha} \\
&= b \left(\sum_{k=-\infty}^{\infty} e^{-3|k|\epsilon} \|L_{f^k(x)}^{-1}\| e^{3(-|k-1|+|k|)\epsilon} \right)^{-1/\alpha}.
\end{aligned}$$

Notice that $-|k-1|+|k|$ can only take the values $+1$ or -1 , depending on the value of k , and therefore

$$b(x)e^{-3\epsilon/\alpha} \leq b \left(\sum_{k=-\infty}^{\infty} e^{-3|k|\epsilon} \|L_{f^k(x)}^{-1}\| e^{3(-|k-1|+|k|)\epsilon} \right)^{-1/\alpha} \leq b(x)e^{3\epsilon/\alpha}.$$

This completes the proof. \square

7. HYPERBOLICITY IN LYAPUNOV CHARTS: THEOREM G

The key motivation for Lyapunov charts is to show that the map f restricted to some neighbourhood of points of Λ is uniformly hyperbolic in Lyapunov coordinates and to study the way f maps such neighbourhoods to each other. To state this precisely, for $x \in \Lambda$ we write

$$\mathcal{B}_x^{s,1} := \Psi_x^{-1}(\mathcal{N}_x \cap f^{-1}\mathcal{N}_{f(x)}) \quad \text{and} \quad \mathcal{B}_{f(x)}^{u,1} := \Psi_{f(x)}^{-1}(f(\mathcal{N}_x) \cap \mathcal{N}_{f(x)})$$

and denote by

$$f_x := \Psi_{f(x)}^{-1} \circ f \circ \Psi_x : \mathcal{B}_x^{s,1} \rightarrow \mathcal{B}_{f(x)}^{u,1}$$

the corresponding diffeomorphism. Similarly, if $x \in \Lambda_{\ell_0}$, and $f(x) \in \Lambda_{\ell_1}$ for some ℓ_1 with $|\ell_1 - \ell_0| \leq 1$ we let $\bar{\ell} = (\ell_0, \ell_1)$ and write

$$(7.1) \quad \mathcal{B}_{x,\bar{\ell}}^{s,1} := \Psi_x^{-1}(\mathcal{N}_x^{(\ell_0)} \cap f^{-1}\mathcal{N}_{f(x)}^{(\ell_1)}) \quad \text{and} \quad \mathcal{B}_{f(x),\bar{\ell}}^{u,1} := \Psi_{f(x)}^{-1}(f\mathcal{N}_x^{(\ell_0)} \cap \mathcal{N}_{f(x)}^{(\ell_1)})$$

By (5.7), $\mathcal{B}_{x,\bar{\ell}}^{s,1} \subseteq \mathcal{B}_x^{s,1}$ and $\mathcal{B}_{f(x),\bar{\ell}}^{u,1} \subseteq \mathcal{B}_{f(x)}^{u,1}$ and therefore f_x restricts to a map

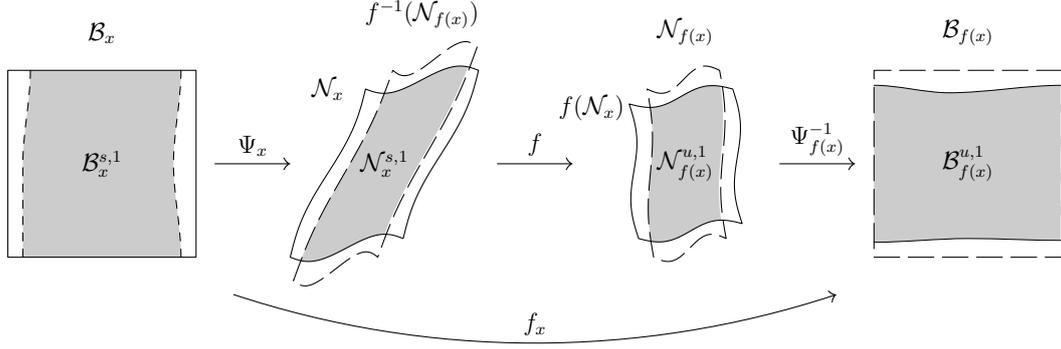
$$f_x : \mathcal{B}_{x,\bar{\ell}}^{s,1} \rightarrow \mathcal{B}_{f(x),\bar{\ell}}^{u,1}$$

For simplicity we will just use the same notation f_x in both cases. We also mention that in the definitions of the sets $\mathcal{B}_{x,\bar{\ell}}^{s,1}, \mathcal{B}_x^{s,1}, \mathcal{B}_{f(x),\bar{\ell}}^{u,1}, \mathcal{B}_{f(x)}^{u,1}$ we implicitly mean the *connected component*, containing x or $f(x)$ respectively, of these sets which *a priori* may not be connected. The main result of this section states that the map f_x is uniformly hyperbolic and the sets just defined have a certain specific geometry, see Figure 7.1.

A key part of the statement of Theorem G below is that certain sets are stable and unstable strips which are *strictly* contained in the sets $\mathcal{B}_x^{(\ell)}$. It is convenient to introduce the following sets.

Definition 7.1. Given $r > 0$, consider the sets

$$\begin{aligned}
\tilde{\mathcal{B}}_r^s &:= [-e^{-\lambda/3}r, e^{-\lambda/3}r] \times [-r, r] \subset \mathbb{R}^2, \\
\tilde{\mathcal{B}}_r^u &:= [-r, r] \times [-e^{-\lambda/3}r, e^{-\lambda/3}r] \subset \mathbb{R}^2
\end{aligned}$$

FIGURE 7.1. The map f in Lyapunov coordinates.

and let

$$(7.2) \quad \tilde{\mathcal{N}}_x^{s/u} = \Psi_x(\tilde{\mathcal{B}}_{b(x)}^{s/u}) \quad \text{and} \quad \tilde{\mathcal{N}}_{x,\ell}^{s/u} = \Psi_x(\tilde{\mathcal{B}}_{b_\ell}^{s/u}).$$

Theorem G. *Let $f: M \rightarrow M$ be a $C^{1+\alpha}$ surface diffeomorphism. Fix $\chi > \lambda > 0$, $0 < \epsilon < \epsilon_1(f, \chi, \lambda)$, and ω satisfying (5.9). Then there is $b > 0$ such that for every (χ, ϵ) -hyperbolic set Λ , $\bar{\ell} = (\ell_0, \ell_1)$ with $|\ell_0 - \ell_1| \leq 1$ and $x \in \Lambda_{\ell_0}$ with $f(x) \in \Lambda_{\ell_1}$, the following holds: for every $\underline{y} \in \mathcal{B}_x^{s,1}$, $\underline{z} \in \mathcal{B}_{f(x)}^{u,1}$, $\underline{v}^{u/s} \in K^{u/s}$,*

$$(7.3) \quad \begin{aligned} D_{\underline{y}} f_x(\underline{v}^u) &\in \tilde{K}^u, & \|D_{\underline{y}} f_x(\underline{v}^u)\| &\geq e^{\lambda/2} \|\underline{v}^u\|, \\ D_{\underline{z}} f_x^{-1}(\underline{v}^s) &\in \tilde{K}^s, & \|D_{\underline{z}} f_x^{-1}(\underline{v}^s)\| &\geq e^{\lambda/2} \|\underline{v}^s\|. \end{aligned}$$

Moreover, the sets $\mathcal{B}_{x,\bar{\ell}}^{s,1}, \mathcal{B}_{f(x),\bar{\ell}}^{u,1}$ are strongly stable and unstable strips, satisfying

$$(7.4) \quad \mathcal{B}_{x,\bar{\ell}}^{s,1} \subseteq \tilde{\mathcal{B}}_{b_{\ell_0}}^s \quad \text{and} \quad \mathcal{B}_{f(x),\bar{\ell}}^{u,1} \subseteq \tilde{\mathcal{B}}_{b_{\ell_1}}^u.$$

Remark 7.2. Notice that the estimates (7.3) hold in particular for all $\underline{y} \in \mathcal{B}_{x,\bar{\ell}}^{s,1}$, $\underline{z} \in \mathcal{B}_{f(x),\bar{\ell}}^{u,1}$ but they do not depend on ℓ and give one-step hyperbolicity in the sense that the expansion and contraction is exhibited immediately after one iteration. This is in contrast with the fact that if $x \in \Lambda_\ell$ for large ℓ we have very poor hyperbolicity estimates on the surface, cf. **(H3)** and (1.2). This is of course the effect of the Lyapunov change of coordinates which has controlled, but very large, distortion, and applies to very small neighbourhoods of x when $x \in \Lambda_\ell$ for large ℓ , recall (5.4) and (5.5). The crucial advantage of writing the estimates as in (7.3) is that we can iterate the map any number of times without loss of hyperbolicity. We only need to worry about the effect of the distortion at the beginning and end of any arbitrarily long piece of orbit in order to recover the actual hyperbolicity estimates for the original map f on the surface.

In the rest of this section we prove Theorem G. In §7.1 we establish the derivative estimates (7.3), in §7.2 we use these to prove the invariance property of the cones, and in §7.3 we prove that $\mathcal{B}_{x,\bar{\ell}}^{s,1}, \mathcal{B}_{x,\bar{\ell}}^{u,1}$ are stable and unstable strips and satisfy (7.4).

7.1. Derivative estimates. Here we prove the hyperbolicity estimates (7.3). We start with the special case $\underline{y} = 0$.

Lemma 7.3. *For every $x \in \Lambda$, $\underline{v}^u \in K^u$, $\underline{v}^s \in K^s$,*

$$(7.5) \quad \|D_0 f_x(\underline{v}^u)\| \geq e^{2\lambda/3} \|\underline{v}^u\| \quad \text{and} \quad \|D_0 f_x^{-1}(\underline{v}^s)\| \geq e^{2\lambda/3} \|\underline{v}^s\|.$$

Before proving Lemma 7.3, we set up some notation. Consider the map

$$(7.6) \quad \hat{f}_x := \exp_{f(x)}^{-1} \circ f \circ \exp_x : T_x M \rightarrow T_{f(x)} M.$$

Then since $f_x := \Psi_{f(x)}^{-1} \circ f \circ \Psi_x = L_{f(x)}^{-1} \circ \exp_{f(x)}^{-1} \circ f \circ \exp_x \circ L_x$, we have

$$(7.7) \quad f_x = L_{f(x)}^{-1} \circ \hat{f}_x \circ L_x.$$

Given $v \in T_x M$, we have

$$(7.8) \quad D_v \hat{f}_x = D_{f \circ \exp_x(v)} \exp_{f(x)}^{-1} \circ D_{\exp_x(v)} f \circ D_v \exp_x.$$

Also $f_x : \mathcal{B}_x^{s,1} \rightarrow \mathcal{B}_{f(x)}^{s,1}$ and for every $\underline{y} \in \mathcal{B}_x^{s,1}$, $D_{\underline{y}} f_x = D_{\underline{y}}(L_{f(x)}^{-1} \circ \hat{f}_x \circ L_x)$. Using that $L_x, L_{f(x)}$ are linear maps, we have

$$(7.9) \quad D_{\underline{y}} f_x = L_{f(x)}^{-1} \circ D_{L_x(\underline{y})} \hat{f}_x \circ L_x$$

With this notation we can easily prove Lemma 7.3.

Proof of Lemma 7.3. For $\underline{y} = 0$ we have $L_x(\underline{y}) = L_x(0) = 0$, so (7.9) gives

$$D_0 f_x = L_{f(x)}^{-1} \circ D_0 \hat{f}_x \circ L_x.$$

Now $\exp_x(0) = x$ gives $f \circ \exp_x(0) = f(x)$, and the exponential function is tangent to the identity at 0, i.e., $D_0 \exp_x = \text{Id}$ and $D_{f(x)} \exp_{f(x)}^{-1} = \text{Id}$, so (7.8) implies $D_0 \hat{f}_x = D_x f_x$, which means that we have

$$(7.10) \quad D_0 f_x = L_{f(x)}^{-1} \circ D_x f \circ L_x.$$

Writing $\underline{v}^u = v_1 e_1 + v_2 e_2$, (6.3) and (7.10) give $D_0 f_x(\underline{v}^u) = A_x v_1 e_1 + B_x v_2 e_2$. Since $\underline{v}^u \in K^u$ implies $|v_2| \leq \omega |v_1|$, we conclude that

$$\frac{\|D_0 f_x(\underline{v}^u)\|^2}{\|\underline{v}^u\|^2} = \frac{A_x^2 v_1^2 + B_x^2 v_2^2}{v_1^2 + v_2^2} \geq \frac{A_x^2}{1 + (v_2/v_1)^2} \geq \frac{e^{2\lambda}}{1 + \omega^2} \geq e^{4\lambda/3},$$

where the last two inequalities use (6.4) and (5.9), respectively. This proves the first half of (7.5); the second is similar. \square

Lemma 7.3 implies that the expansion estimates in (7.3) hold in some neighbourhood of $0 \in \mathcal{B}_x^{s,1}$. We show that this neighbourhood contains $\mathcal{B}_x^{s,1}$, which is the key part of the proof of Theorem G. The main step is the following estimate for the derivatives of \hat{f}_x .

Lemma 7.4. *There exists $Q_2 > 0$ such that for all $x \in \Lambda$ and $\underline{y}, \underline{z} \in \mathcal{B}_x^{s,1}$,*

$$\|D_{L_x(\underline{y})} \hat{f}_x - D_{L_x(\underline{z})} \hat{f}_x\| \leq Q_2 \|\underline{y} - \underline{z}\|^\alpha.$$

Proof. For simplicity we write $v := L_x(\underline{y})$, $u := L_x(\underline{z})$. Since L_x is a contraction and $\underline{y}, \underline{z} \in \mathcal{B}_x^{s,1} \subseteq [-b(x), b(x)]^2 \subset [-b, b]^2$, we have $\|v - u\| \leq \|\underline{y} - \underline{z}\| \leq 2\sqrt{2}b$. In particular it is sufficient to prove that $\|D_v \hat{f}_x - D_u \hat{f}_x\| \leq Q \|v - u\|^\alpha$ for all $u, v \in T_x M$ with $\|u\|, \|v\| \leq b$. Since M is a C^2 Riemannian manifold, there is $Q_3 > 0$ such that $\|D_v \exp_x - D_u \exp_x\| \leq Q_3 \|u - v\|$, $\|D_{f \circ \exp_x v} \exp_x^{-1} - D_{f \circ \exp_x u} \exp_x^{-1}\| \leq Q_3 \|u - v\|$ for all $x \in M$ and $u, v \in T_x M$ with $\|u\|, \|v\| \leq 1$. Moreover, there is $Q_4 > 0$ such that $d(\exp_x(u), \exp_x(v)) \leq Q_4 \|u - v\|$ for all such x, u, v , and hence since Df is Hölder continuous we have $\|D_{\exp_x(v)} f - D_{\exp_x(u)} f\| \leq |Df|_\alpha Q_4^\alpha \|u - v\|^\alpha$. Then the definition of $D_v \hat{f}_x, D_u \hat{f}_x$ in (7.8) gives the result. \square

Lemma 7.5. *There exists $Q_5 > 0$ such that for all $x \in \Lambda$ and $\underline{y}, \underline{z} \in \mathcal{B}_x^{s,1}$,*

$$\|D_{\underline{y}} f_x - D_{\underline{z}} f_x\| \leq Q_5 b^\alpha.$$

Proof. By (7.9), using $\|L_x\| \leq 1$ and Lemma 7.4, for every $\underline{y}, \underline{z} \in \mathcal{B}_x^{s,1}$,

$$\begin{aligned} \|D_{\underline{y}} f_x - D_{\underline{z}} f_x\| &= \|L_{f(x)}^{-1} \circ D_{L_x(\underline{y})} \hat{f}_x \circ L_x - L_{f(x)}^{-1} \circ D_{L_x(\underline{z})} \hat{f}_x \circ L_x\| \\ (7.11) \quad &= \|L_{f(x)}^{-1} \circ (D_{L_x(\underline{y})} \hat{f}_x - D_{L_x(\underline{z})} \hat{f}_x) \circ L_x\| \\ &\leq \|L_{f(x)}^{-1}\| \cdot \|D_{L_x(\underline{y})} \hat{f}_x - D_{L_x(\underline{z})} \hat{f}_x\| \\ &\leq Q_2 \|L_{f(x)}^{-1}\| \|\underline{y} - \underline{z}\|^\alpha \end{aligned}$$

Moreover, $\underline{y}, \underline{z} \in \mathcal{B}_x^{s,1} \subset [-b(x), b(x)]^2$ implies $\|\underline{y} - \underline{z}\| \leq 2b(x)$ and therefore, by Lemma 6.5,

$$(7.12) \quad \|\underline{y} - \underline{z}\|^\alpha \leq 2^\alpha b(x)^\alpha \leq 2^\alpha e^{3\epsilon} b(f(x))^\alpha.$$

Moreover, from (5.5) we have

$$b(f(x))^\alpha := b^\alpha \left(\sum_{k=-\infty}^{\infty} e^{-3|k|\epsilon} \|L_{f^{k+1}(x)}^{-1}\| \right)^{-1} \leq b^\alpha \|L_{f(x)}^{-1}\|^{-1}$$

and therefore, substituting into (7.12) and then into (7.11) we get

$$\|D_{\underline{y}} f_x - D_{\underline{z}} f_x\| \leq Q_2 \|L_{f(x)}^{-1}\| \|\underline{y} - \underline{z}\|^\alpha \leq 2^\alpha e^{3\epsilon} Q_2 b^\alpha,$$

which completes the proof. \square

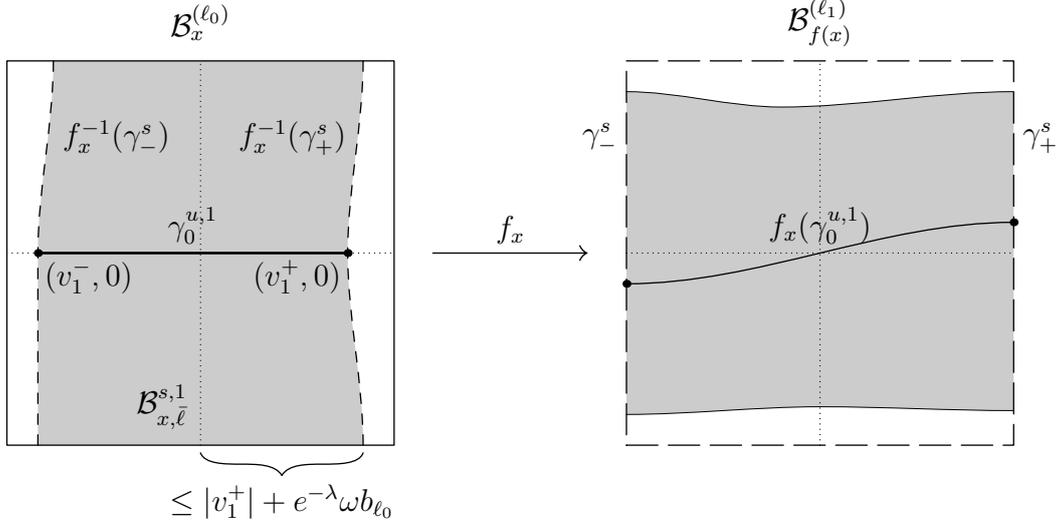
Now we can prove the expansion estimates in (7.3) for all \underline{y} . By Lemmas 7.3 and 7.5, for every $x \in \Lambda$, $\underline{y} \in \mathcal{B}_x^{s,1}$, and $\underline{v}^u \in K^u$, we have

$$\|D_{\underline{y}} f_x(\underline{v}^u)\| \geq \|D_0 f_x(\underline{v}^u)\| - \|D_{\underline{y}} f_x - D_0 f_x\| \cdot \|\underline{v}^u\| \geq (e^{2\lambda/3} - Q_5 b^\alpha) \|\underline{v}^u\|.$$

Choose $b > 0$ small enough that

$$(7.13) \quad e^{2\lambda/3} - Q_5 b^\alpha \geq e^{\lambda/2};$$

then we get $\|D_{\underline{y}} f_x(\underline{v}^u)\| \geq e^{\lambda/2} \|\underline{v}^u\|$. A similar argument gives $\|D_{\underline{y}} f_{f(x)}^{-1}(\underline{v}^s)\| \geq e^{\lambda/2} \|\underline{v}^s\|$ for every $\underline{v}^s \in K^s$, and so we have the expansion estimates in (7.3).


 FIGURE 7.2. $\mathcal{B}_x^{s,1}$ is a strongly stable strip.

7.2. Conefield invariance. We now prove the conefield invariance from (7.3). Fix $\eta > 0$ small enough that if $z = z_1 e_1 + z_2 e_2 \in \mathbb{R}^2$ has $\|z\| = 1$ and $z_2 < e^{-2\lambda} \omega z_1$, then every $v \in \mathbb{R}^2$ with $\|v - z\| < \eta$ is contained in \tilde{K}^u . By homogeneity we see that if the assumption on $\|z\|$ is removed and we have $\|v - z\| < \eta \|z\|$, then once again $v \in \tilde{K}^u$. Given $x \in \Lambda$ and $\underline{v} = v_1 e_1 + v_2 e_2 \in K^u$, (6.3) gives $D_0 f_x(\underline{v}) = A_x v_1 e_1 + B_x v_2 e_2$, and we have $|B_x v_2| < e^{-\lambda} \omega |v_1| < e^{-2\lambda} \omega |A_x v_1|$, so $D_0 f_x(\underline{v})$ satisfies the assumption on z mentioned above. Now choose b small enough that

$$(7.14) \quad Q_5 b^\alpha < \eta.$$

Then for every $\underline{y} = \Psi_x^{-1}(y) \in \mathcal{B}_x^{s,1}$, Lemma 7.5 gives $\|D_{\underline{y}} f_x(\underline{v}) - D_0 f_x(\underline{v})\| \leq Q_5 b^\alpha \|\underline{v}\| < \eta \|\underline{v}\|$, and by our choice of η we conclude that $D_{\underline{y}} f_x(\underline{v}) \in \tilde{K}^u$. A completely symmetric argument applies to the stable cones and f^{-1} .

7.3. Stable and unstable strips. To complete the proof of Theorem G we show that $\mathcal{B}_{x, \bar{\ell}}^{s,1}$ and $\mathcal{B}_{f(x), \bar{\ell}}^{u,1}$ are strongly stable and unstable strips in $\tilde{\mathcal{B}}_{b_{\ell_0}}^s, \tilde{\mathcal{B}}_{b_{\ell_1}}^u$ respectively, recall (7.1) and (7.4). We begin by proving the statement for $\mathcal{B}_{x, \bar{\ell}}^{s,1}$. Let $\gamma_0^{u,1} := \{(v_1, 0) \in \mathcal{B}_x^{(\ell_0)} : f_x(v_1, 0) \in \mathcal{B}_{f(x)}^{(\ell_1)}\} \subseteq \mathcal{B}_{x, \bar{\ell}}^{s,1}$. Notice that $f_x(0) = 0$ and therefore $\mathcal{B}_{x, \bar{\ell}}^{s,1}$ contains a neighbourhood of 0 and therefore $\gamma_0^{u,1}$ is a non-trivial horizontal segment, and in particular its tangent vectors are contained in the unstable cones K^u . Therefore, by (7.3), the images of the tangent vectors to $\gamma_0^{u,1}$ are contained in the strong unstable cones \tilde{K}^u and in particular the slope of the curve $f_x(\gamma_0^{u,1})$ always has absolute value $< e^{-\lambda} \omega < 1$.

Since $f_x(\gamma_0^{u,1})$ goes through the origin, has slope < 1 in absolute value, and has both endpoints on the boundary of the square $\mathcal{B}_{f(x)}^{(\ell_1)} = [-b_{\ell_1}, b_{\ell_1}]^2$ these endpoints must both lie on the stable boundaries $\gamma_{\pm}^s := \{(\pm b_{\ell_1}, v_2), v_2 \in$

$[-b_{\ell_1}, b_{\ell_1}]$ of $\mathcal{B}_{f(x)}^{(\ell_1)}$, so $f_x(\gamma_0^{u,1})$ is a *full length* strongly unstable curve in $\mathcal{B}_{f(x)}^{(\ell_1)}$ as shown in Figure 7.2. Therefore the preimages $f_x^{-1}(\gamma_{\pm}^s) \cap \mathcal{B}_x^{(\ell_0)}$ are, by (7.3), strongly stable curves through the endpoints of $\gamma_0^{u,1}$, which are points on the horizontal axis with coordinates $(v_1^{\pm}, 0)$ with $|v_1^{\pm}| \leq e^{-\lambda/2} b_{\ell_1}$. Note that this last estimate comes from the fact that every tangent vector to $f_x(\gamma_0^{u,1})$ is contracted by a factor of at least $e^{-\lambda/2}$ under the action of f_x^{-1} . Since $f_x^{-1}(\gamma_{\pm}^s)$ are strongly stable curves in $\mathcal{B}_x^{(\ell_0)}$, we conclude as shown in Figure 7.2 that their horizontal distance from the y -axis in \mathbb{R}^2 is always bounded above by

$$|v_1^{\pm}| + e^{-\lambda} \omega b_{\ell_0} \leq e^{-\lambda/2} b_{\ell_1} + e^{-\lambda} \omega b_{\ell_0} < (e^{-\lambda/2} e^{3\epsilon/\alpha} + e^{-\lambda} \omega) b_{\ell_0},$$

where the last inequality uses Lemma 6.5. By (5.9), this is $< e^{-\lambda/3} b_{\ell_0}$, which proves that $\mathcal{B}_{x,\bar{\ell}}^{s,1}$ is a strongly stable strip in $\tilde{\mathcal{B}}_{b_{\ell_0}}^s$. Similarly, $\mathcal{B}_{f(x),\bar{\ell}}^{u,1}$ is a strongly unstable strip in $\tilde{\mathcal{B}}_{b_{\ell_1}}^u$.

8. OVERLAPPING CHARTS: THEOREM H

8.1. Overlapping charts. The parameters defining Lyapunov charts vary slowly along orbits but in general only measurably with the point $x \in \Lambda$. On each regular level set Λ_{ℓ} , the dependence is continuous, and it is well-known that “if points $x, y \in \Lambda_{\ell}$ are close, then their Lyapunov charts are close”. The condition on how close x, y need to be depends on ℓ ; we need an explicit quantitative estimate, which is provided by (8.3) in Theorem H below. This is the core technical result of the paper, whose proof demands the largest share of our efforts.

First we make precise what it means for two Lyapunov charts to be close. Let $\chi > \lambda > 0$ be fixed, ϵ_1 given by (1.7), and $\epsilon \in (0, \epsilon_1)$. Let Λ be a (χ, ϵ) -hyperbolic set. Given $\ell \geq 1$ and $x, y \in \Lambda_{\ell}$, recall that $\mathcal{N}_x^{(\ell)}, \mathcal{N}_y^{(\ell)}$ are defined in (5.8) and $\tilde{\mathcal{N}}_{x,\ell}^{s/u}, \tilde{\mathcal{N}}_{y,\ell}^{s/u}$ in (7.2).

Definition 8.1 (Overlapping charts). We say that $\mathcal{N}_x^{(\ell)}$ and $\mathcal{N}_y^{(\ell)}$ are *overlapping* if $x, y \in \mathcal{N}_x^{(\ell)} \cap \mathcal{N}_y^{(\ell)}$ and the following conditions hold:

A) Overlapping derivative estimates: for every $\underline{z} \in \Psi_x^{-1}(\mathcal{N}_x^{(\ell)} \cap \mathcal{N}_y^{(\ell)})$ and every $\underline{v}^u \in \tilde{K}^u, \underline{v}^s \in \tilde{K}^s$, we have

$$(8.1) \quad D_{\underline{z}}(\Psi_y^{-1} \circ \Psi_x)(\underline{v}^u) \subset K^u \quad \text{and} \quad \|D_{\underline{z}}(\Psi_y^{-1} \circ \Psi_x)(\underline{v}^u)\| \geq e^{-\lambda/24} \|\underline{v}^u\|,$$

$$(8.2) \quad D_{\underline{z}}(\Psi_x^{-1} \circ \Psi_y)(\underline{v}^s) \subset K^s \quad \text{and} \quad \|D_{\underline{z}}(\Psi_x^{-1} \circ \Psi_y)(\underline{v}^s)\| \geq e^{-\lambda/24} \|\underline{v}^s\|,$$

and similarly with the roles of x and y reversed.

B) Overlapping stable and unstable strips: Every full length strongly stable curve $\gamma^s \subset \tilde{\mathcal{N}}_{x,\ell}^s$ (resp. $\tilde{\mathcal{N}}_{y,\ell}^s$) completely crosses $\tilde{\mathcal{N}}_{y,\ell}^u$ (resp. $\tilde{\mathcal{N}}_{x,\ell}^u$) and every full length strongly unstable curve $\gamma^u \subset \tilde{\mathcal{N}}_{x,\ell}^u$ (resp. $\tilde{\mathcal{N}}_{y,\ell}^u$) completely crosses $\tilde{\mathcal{N}}_{y,\ell}^s$ (resp. $\tilde{\mathcal{N}}_{x,\ell}^s$).

Theorem H. *Let f be a $C^{1+\alpha}$ surface diffeomorphism. For every $\chi > \lambda > 0$, and every $0 < \epsilon < \epsilon_1(f, \chi, \lambda)$, there exists $\delta > 0$ such that given any (χ, ϵ) -hyperbolic set Λ , any integer $\ell \in \mathbb{N}$, and $x, y \in \Lambda_\ell$ with*

$$(8.3) \quad d(x, y) \leq \delta e^{-\lambda \ell},$$

the Lyapunov charts $\mathcal{N}_x^{(\ell)}$ and $\mathcal{N}_y^{(\ell)}$ are overlapping.

In the rest of this section we prove Theorem H. The proof depends on two intermediate results about the Hölder continuity of $E_x^{s/u}$ and of $s(x), u(x)$ which we prove in §8.2 and 8.3 respectively. In §8.4 and §8.5 we combine these to prove parts A) and B) respectively in the definition of overlapping charts, thus completing the proof of Theorem H.

8.2. Hölder continuity of the splitting.

Proposition 8.2. *There is $Q_6 > 0$ such that for any $\ell \in \mathbb{N}$ and $x, y \in \Lambda_\ell$, we have*

$$d(E_x^s, E_y^s) \leq Q_6 e^{6\epsilon\gamma\ell} d(x, y)^\beta \quad \text{and} \quad d(E_x^u, E_y^u) \leq Q_6 e^{6\epsilon\gamma\ell} d(x, y)^\beta$$

where $d(\cdot, \cdot)$ represents distance in the Grassmannian of M .

Recall that β is given in (1.6). For generality we prove Proposition 8.2 as a special case of the following result which does not require x, y to belong to a (χ, ϵ) -hyperbolic set. More specifically the Hölder continuity only depends on the angle and hyperbolicity estimates at the points x, y and not on how these vary along the orbits of x, y .

Proposition 8.3. *Let M be a compact smooth Riemannian manifold and $f: M \rightarrow M$ a $C^{1+\alpha}$ diffeomorphism. There is a constant Q_6 , depending only on M , $\|Df^{\pm 1}\|$, α , $|Df^{\pm 1}|_\alpha$, and χ , such that if $C, K > 0$ and $x, y \in M$ are such that*

$$(8.4) \quad \|Df_x^n e_x^s\| \leq C e^{-\chi n} \quad \text{and} \quad \|Df_y^n e_y^s\| \leq C e^{-\chi n} \quad \text{for all } n \geq 0,$$

$$(8.5) \quad \|Df_x^n e_x^u\| \geq C^{-1} e^{\chi n} \quad \text{and} \quad \|Df_y^n e_y^u\| \geq C^{-1} e^{\chi n} \quad \text{for all } n \geq 0,$$

for some unit vectors $e_x^{s/u} \in T_x M$, $e_y^{s/u} \in T_y M$ for which the corresponding subspaces $E_{x/y}^{s/u}$ satisfy $\angle(E_x^s, E_x^u) \geq K$, $\angle(E_y^s, E_y^u) \geq K$, then we have

$$(8.6) \quad d(E_x^s, E_y^s) \leq Q_6 (C^2 K^{-1})^{2\gamma} d(x, y)^\beta,$$

where γ, β are as in (1.6). The same bound holds for E_x^u, E_y^u if we have

$$(8.7) \quad \|Df_x^{-n} e_x^u\| \leq C e^{-\chi n} \quad \text{and} \quad \|Df_y^{-n} e_y^u\| \leq C e^{-\chi n} \quad \text{for all } n \geq 0,$$

$$(8.8) \quad \|Df_x^{-n} e_x^s\| \geq C^{-1} e^{\chi n} \quad \text{and} \quad \|Df_y^{-n} e_y^s\| \geq C^{-1} e^{\chi n} \quad \text{for all } n \geq 0.$$

In particular, if (8.4), (8.5), (8.7), and (8.8) all hold, then

$$(8.9) \quad |\angle(E_x^s, E_x^u) - \angle(E_y^s, E_y^u)| \leq Q_6 (C^2 K^{-1})^{2\gamma} d(x, y)^\beta.$$

To see that Proposition 8.3 implies Proposition 8.2, it suffices to observe that given $\epsilon > 0$ and $\ell \in \mathbb{N}$, the conditions (8.4), (8.5), (8.7), (8.8) are satisfied for all $x, y \in \Lambda_\ell$, with $C = e^{\epsilon\ell}$ and $K = e^{-\epsilon\ell}$.

We will give the explicit calculations only for the stable subspaces, leading to the proof of (8.6), as the situation for the unstable subspaces is completely symmetrical. We follow Brin's approach in [12, Appendix A] (see also [13, §5.3]; the main idea of the argument goes back to [17]). The only thing we need that is not given there is the computation for how much vectors in $(E_x^s)^\perp$ are expanded, depending on C and K , given in Lemma 8.7 below. The rest of the proof of Proposition 8.3 is taken nearly verbatim from [12, Appendix A], with notation adjusted to fit our current setting.

First note that by the Whitney embedding theorem [37], we can choose $N \in \mathbb{N}$ such that M can be smoothly embedded in \mathbb{R}^N . By compactness of M , the Riemannian metric is uniformly equivalent to the distance induced by the embedding and therefore it suffices to prove the result under the assumption that $M \subset \mathbb{R}^N$. Then, for each $x \in M$ write $E^\perp(x)$ for the orthogonal complement to $T_x M \subset \mathbb{R}^N$; since E^\perp is smooth it suffices to prove the result with E_x^s replaced by $\tilde{E}_x^s := E_x^s \oplus E^\perp(x)$.

Definition 8.4. Given $x \in M \subset \mathbb{R}^N$ and $n \in \mathbb{N}$, let $D_x^{(n)}$ be the $N \times N$ matrix representing the linear map that takes $v \mapsto D_x f^n(v)$ for $v \in T_x M$ and $v \mapsto 0$ for $v \in E^\perp(x)$.

Since we embed M in \mathbb{R}^N , we can treat Grassmannian distance between subspaces as follows. Given a subspace $E \subset \mathbb{R}^N$, we define the distance of a non-zero vector v from the subspace E by considering the unique decomposition $v = v^E + v^\perp$ where $v^E \in E$ and $v^\perp \perp E$ and letting $d(v, E) := \|v^\perp\|/\|v\|$. We can then define the distance between two subspaces $E, E' \subset \mathbb{R}^N$ by

$$(8.10) \quad d(E, E') := \sup\{d(v, E) : v \in E' \setminus \{0\}\} = \sup\{d(v, E') : v \in E \setminus \{0\}\}.$$

The strategy of the proof is based on the following general result.

Lemma 8.5. [12, Lemma A.1] *Let $N \geq 2$ and let $\{A_k\}, \{B_k\}$, be two sequences of real $N \times N$ matrices satisfying the following properties*

(1) *there are $\Delta \in (0, 1)$ and $c_3 > 0$ such that*

$$(8.11) \quad \|A_k - B_k\| \leq \Delta e^{c_3 k} \text{ for all } k \geq 0;$$

(2) *there are subspaces $E_A, E_B \subset \mathbb{R}^N$, $\chi > 0$, and $C' > 1$ such that*

$$(8.12) \quad \|A_k v_A\| \leq C' e^{-\chi k} \|v_A\| \quad \text{and} \quad \|A_k v_A^\perp\| \geq (C')^{-1} e^{\chi k} \|v_A^\perp\|$$

for every $v_A \in E_A, v_A^\perp \perp E_A, k \geq 0$, and

$$(8.13) \quad \|B_k v_B\| \leq C' e^{-\chi k} \|v_B\| \quad \text{and} \quad \|B_k v_B^\perp\| \geq (C')^{-1} e^{\chi k} \|v_B^\perp\|$$

for every $v_B \in E_B, v_B^\perp \perp E_B, k \geq 0$.

Then

$$(8.14) \quad d(E_A, E_B) \leq 3(C')^2 e^{2\chi} \Delta^{\frac{2\chi}{c_3 + \chi}}.$$

Proof. We start by fixing $q := -(\chi + c_3)$, and let $k_0 := \lfloor \frac{\log \Delta}{q} \rfloor$, so that

$$(k_0 + 1)q < \log \Delta \leq k_0 q \leq \log \Delta - q$$

(recall that $q < 0$). In particular,

$$(8.15) \quad \Delta e^{c_3 k_0} \leq e^{k_0 q} e^{a k_0} = e^{-\chi k_0}$$

and, letting $\xi := \frac{2\chi}{c_3 + \chi} = -\frac{2\chi}{q}$,

$$(8.16) \quad e^{-2\chi k_0} = (e^{q k_0})^{-\frac{2\chi}{q}} = (e^{q k_0})^\xi \leq e^{-q\xi} \Delta^\xi = e^{2\chi} \Delta^\xi,$$

where the inequality uses $k_0 q < -q + \log \Delta$. Then, by (8.11), for every $v_B \in E_B$ and every $k \geq 1$ we have

$$\|A_k v_B\| \leq \|B_k v_B\| + \|A_k - B_k\| \cdot \|w\| \leq C' e^{-\chi k} \|v_B\| + \Delta e^{c_3 k} \|v_B\|$$

and therefore, in particular, for $k = k_0$, by (8.15),

$$\|A_{k_0} v_B\| \leq C' e^{-\chi k_0} \|v_B\| + \Delta e^{c_3 k_0} \|v_B\| \leq 2C' e^{-\chi k_0} \|v_B\|.$$

This implies that

$$E_B \subset R_A := \{v \in \mathbb{R}^N : \|A_{k_0} v\| \leq 2C' e^{-\chi k_0} \|v\|\}.$$

Clearly we also have $E_A \subset R_A$ and therefore it is sufficient to estimate the “width” of R_A . For $v \in R_A$, write $v = v_A + v_A^\perp$, where $v_A \in E_A$ and $v_A^\perp \perp E_A$. Then by (8.12), for any $k \geq 1$ we have

$$\|A_k v\| \geq \|A_k v_A^\perp\| - \|A_k v_A\| \geq (C')^{-1} e^{\chi k} \|v_A^\perp\| - C' e^{-\chi k} \|v_A\|,$$

and therefore, for $k = k_0$, using also that $\|v_A\| \leq \|v\|$ since the the splitting of v is orthogonal, we get

$$\|v_A^\perp\| \leq C' e^{-\chi k_0} (\|A_{k_0} v\| + C' e^{-\chi k_0} \|v_A\|) \leq 3(C')^2 e^{-2\chi k_0} \|v\|,$$

which, by (8.16), implies $d(v, E_A) \leq 3(C')^2 e^{-2\chi k_0} \leq 3(C')^2 e^{2\chi} \Delta^\xi$ and therefore, from the definition of ξ , the conclusions of the Lemma. \square

The following two Lemmas give the estimates we need to apply Lemma 8.5. Recall that c_1, c_2, c_3 are as in (1.5), that M is embedded in \mathbb{R}^N , and that $D_x^{(n)}$ are the matrices defined in Definition 8.4.

Lemma 8.6. [12, Lemma A.2] *There is $Q_7 \geq 1$ such that for all $x, y \in M$ and every $n \geq 1$, we have*

$$\|D_x^{(n)} - D_y^{(n)}\| \leq Q_7 e^{c_3 n} \|x - y\|^\alpha.$$

Proof. We prove the Lemma by induction on n . For $n = 1$, since f is $C^{1+\alpha}$ we have $\|D_x^{(1)} - D_y^{(1)}\| \leq |Df|_\alpha \|x - y\|^\alpha$ and therefore the statement holds for any $Q_7 \geq |Df|_\alpha$. Then, by the chain rule, we have

$$D_x^{(n+1)} - D_y^{(n+1)} = D_{f^n(x)}^{(1)} D_x^{(n)} - D_{f^n(y)}^{(1)} D_y^{(n)};$$

by adding and subtracting $D_{f^n(x)}^{(1)} D_y^{(n)}$, taking norms, using the inductive assumption and the fact that $\|f^n x - f^n y\| \leq e^{c_2 n} \|x - y\|$ for all $x, y \in M$, we get

$$\|D_x^{(n+1)} - D_y^{(n+1)}\| \leq \|D_{f^n(x)}^{(1)}\| \cdot \|D_x^{(n)} - D_y^{(n)}\| + \|D_{f^n(x)}^{(1)} - D_{f^n(y)}^{(1)}\| \cdot \|D_y^{(n)}\|$$

$$\begin{aligned} &\leq e^{c_2} Q_7 e^{c_3 n} \|x - y\|^\alpha + |Df|_\alpha e^{c_2 n \alpha} \|x - y\|^\alpha e^{c_2 n} \\ &\leq Q_7 e^{c_3(n+1)} \|x - y\|^\alpha (e^{c_2 - c_3} + |Df|_\alpha Q_7^{-1} e^{(1+\alpha)c_2 n} e^{-c_3(n+1)}). \end{aligned}$$

Since by (1.5), $c_3 > (1 + \alpha)c_2$, we can choose Q_7 sufficiently large so that the quantity inside the brackets is less than 1 for every n , which completes the proof. \square

Lemma 8.7. *Suppose A_k is a sequence of $N \times N$ matrices and $\mathbb{R}^N = E^s \oplus E^u$ is a splitting such that $\angle(E^s, E^u) \geq K > 0$, and for every $k \geq 1$ and $v^u \in E^u$, $v^s \in E^s$ we have*

$$(8.17) \quad \|A_k v^s\| \leq C e^{-\chi k} \|v^s\| \quad \text{and} \quad \|A_k v^u\| \geq C^{-1} e^{\chi k} \|v^u\|.$$

Then for every $w \perp E^s$ and every $k \geq 1$, we have

$$(8.18) \quad \|A_k w\| \geq (2C^2 K^{-1})^\gamma e^{\chi k} \|w\|,$$

where $\gamma := \frac{\chi - c_1}{2\chi}$ as in (1.6).

Proof. Writing $w = w^u + w^s$ where $w^u \in E^u$, $w^s \in E^s$, from (8.17) we get

$$\|A_k w\| \geq \|A_k w^u\| - \|A_k w^s\| \geq C^{-1} e^{\chi k} \|w^u\| - C e^{-\chi k} \|w^s\|.$$

Let $\theta = \angle(w^s, w^u)$ and note that $\theta \geq K$ and since $w \perp w^s$, we have $\|w\| = \|w^u\| \sin \theta \leq \|w^u\|$ and $\|w\| = \|w^s\| \tan \theta \geq \|w^s\| \tan K \geq \|w^s\| K$. Plugging this into the equation above, gives

$$(8.19) \quad \|D_x f^k w\| \geq (C^{-1} e^{\chi k} - C K^{-1} e^{-\chi k}) \|w\|.$$

Now fix $k_0 := \lfloor (2\chi)^{-1} \log(2C^2 K^{-1}) \rfloor$, Then for $k \leq k_0$ we have

$$\frac{\|A_k w\|}{e^{\chi k} \|w\|} \geq \frac{e^{c_1 k}}{e^{\chi k}} \geq e^{(c_1 - \chi)k_0} \geq e^{(c_1 - \chi)(2\chi)^{-1} \log(2C^2 K^{-1})} = (2C^2 K^{-1})^{\frac{c_1 - \chi}{2\chi}}$$

where we recall that $c_1 < 0$ (see (1.5)). The formula for γ gives the required estimate.

It remains to treat $k > k_0$. In this case we have

$$e^{-2\chi k} \leq e^{-2\chi(k_0+1)} \leq e^{-\log(2C^2 K^{-1})} = \frac{1}{2} C^{-2} K,$$

which gives

$$C K^{-1} e^{-\chi k} \leq \frac{1}{2} C^{-1} e^{\chi k}$$

and hence, (8.19) gives

$$\|A_k w\| \geq (2C)^{-1} e^{\chi k} \|w\|.$$

Since $\gamma > 1$, $C \geq 1$, and $K \leq 1$, we have

$$(2C^2 K^{-1})^{-\gamma} \leq (2C^2 K^{-1})^{-1} = (2C)^{-1} (C K^{-1})^{-1} \leq (2C)^{-1},$$

and thus we get the result in this case also, thus completing the proof. \square

To complete the proof of Proposition 8.3, we apply Lemma 8.5 with

$$A_k = D_x^{(k)}, \quad B_k = D_y^{(k)}, \quad \Delta = Q_7 \|x - y\|^\alpha, \quad C' = (2C^2 K^{-1})^\gamma.$$

Lemma 8.6 shows that (8.11) holds, while (8.12) and (8.13) follow from (8.4), (8.5), and Lemma 8.7. Thus Lemma 8.5 applies, and using (1.6) to write $\frac{2\chi}{c_3 + \chi} = \frac{\beta}{\alpha}$, we have

$$(8.20) \quad d(E_x^s, E_y^s) \leq 3(C')^2 e^{2\chi} \Delta^{\frac{\beta}{\alpha}} = 3(2C^2 K^{-1})^{2\gamma} e^{2\chi} (Q_7 d(x, y)^\alpha)^{\frac{\beta}{\alpha}},$$

which completes the proof of Proposition 8.3.

8.3. Hölder continuity of Lyapunov coordinates. In this section we prove that $s, u: \Lambda_\ell \rightarrow [\sqrt{2}, Q_0 \tilde{Q}^{-1} e^{\epsilon\ell}]$ are Hölder continuous with exponent ζ and constant given in terms of $e^{\epsilon\eta\ell}$, where ζ, η are given in (1.6). Observe that the Hölder exponent ζ depends on $\chi - \lambda$, and decays to 0 as $\lambda \rightarrow \chi$ where χ is the decay rate associated to the (χ, ϵ) -hyperbolic set Λ and $\lambda < \chi$ is the rate used in the definition of $s(x), u(x)$.

Proposition 8.8. *There is $Q_8 > 0$ such that for any $\ell \in \mathbb{N}$ and $x, y \in \Lambda_\ell$, we have*

$$(8.21) \quad |s(x) - s(y)| \leq Q_8 e^{\epsilon\eta\ell} d(x, y)^\zeta \quad \text{and} \quad |u(x) - u(y)| \leq Q_8 e^{\epsilon\eta\ell} d(x, y)^\zeta.$$

We give the argument for the upper bound for $|s(x) - s(y)|$; the argument for $|u(x) - u(y)|$ is analogous. Recall first that by definition,

$$s(x)^2 = 2 \sum_{n \geq 0} e^{2\lambda n} \|Df_x^n e_x^s\|^2.$$

Since $s(x), s(y) \geq 1$, we have

$$(8.22) \quad |s(x) - s(y)| \leq \frac{|s(x)^2 - s(y)^2|}{2} \leq \sum_{n \geq 0} e^{2\lambda n} \left| \|Df_x^n e_x^s\|^2 - \|Df_y^n e_y^s\|^2 \right|.$$

Notice that $x, y \in \Lambda_\ell$ gives $\|Df_x^n e_x^s\| \leq e^{\epsilon\ell} e^{-\chi n}$, $\|Df_y^n e_y^s\| \leq e^{\epsilon\ell} e^{-\chi n}$, and so

$$(8.23) \quad \Delta_n = \Delta_n(x, y) := \left| \|Df_x^n e_x^s\|^2 - \|Df_y^n e_y^s\|^2 \right| \leq 2e^{2\epsilon\ell} e^{-2\chi n}.$$

Plugging this into (8.22) gives a uniform bound for $|s(x) - s(y)|$ but is not sufficient for our purposes since it does not include $d(x, y)$ and does not therefore imply Hölder continuity. It will nevertheless be useful to bound the tail of the sum for large values of n . For small n we need a more sophisticated estimate on Δ_n , as follows.

Lemma 8.9. *There is $Q_9 > 0$ such that for all $\ell \in \mathbb{N}$ and $x, y \in \Lambda_\ell$ we have*

$$(8.24) \quad \Delta_n \leq Q_9 e^{(2+\alpha\beta)c_2 n} e^{6\epsilon\gamma\alpha(\ell+n)} d(x, y)^{\alpha\beta}.$$

The proof of Lemma 8.9 uses the Hölder continuity of the hyperbolic splitting from Proposition 8.2 and so for clarity we isolate the specific estimate in which this property is used.

Sublemma 8.10. *There is $Q_{10} > 0$ such that for all $\ell \in \mathbb{N}$, $x, y \in \Lambda_\ell$, and $k \geq 0$, we have*

$$(8.25) \quad \left| \|Df_{f^k x} e_{f^k x}^s\| - \|Df_{f^k y} e_{f^k y}^s\| \right| \leq Q_{10} e^{6\epsilon\gamma\alpha(\ell+k)} d(f^k x, f^k y)^{\alpha\beta}.$$

Proof. Since Df is Hölder on TM we have

$$\left| \|Df_{f^k x} e_{f^k x}^s\| - \|Df_{f^k y} e_{f^k y}^s\| \right| \leq |Df|_\alpha d(e_{f^k x}^s, e_{f^k y}^s)^\alpha$$

By (1.3), $f^k x, f^k y \in \Lambda_{\ell+k}$ and therefore, by Proposition 8.2,

$$d(e_{f^k x}^s, e_{f^k y}^s)^\alpha \leq |Df|_\alpha (Q_6 e^{6\epsilon\gamma(\ell+k)} d(f^k x, f^k y)^\beta)^\alpha$$

which gives the result. \square

Proof of Lemma 8.9. By (1.5) the norm of $\|Df\|$ is bounded above by e^{c_2} , and therefore, using the formula for the difference of two squares,

$$(8.26) \quad \Delta_n \leq 2e^{c_2 n} \left| \|Df_x^n e_x^s\| - \|Df_y^n e_y^s\| \right|.$$

Moreover, by the chain rule we have

$$(8.27) \quad \left| \|Df_x^n e_x^s\| - \|Df_y^n e_y^s\| \right| = \left| \prod_{k=0}^{n-1} \|Df_{f^k x} e_{f^k x}^s\| - \prod_{k=0}^{n-1} \|Df_{f^k y} e_{f^k y}^s\| \right|$$

and therefore, applying the standard equality for the difference of two products $|\prod_{k=0}^{n-1} a_k - \prod_{k=0}^{n-1} b_k| = |\sum_{k=0}^{n-1} a_0 \dots a_{k-1} (a_k - b_k) b_{k+1} \dots b_{n-1}|$, and using that the absolute value of each individual term is bounded by e^{c_2} , we get

$$\left| \prod_{k=0}^{n-1} \|Df_{f^k x} e_{f^k x}^s\| - \prod_{k=0}^{n-1} \|Df_{f^k y} e_{f^k y}^s\| \right| \leq e^{c_2 n} \sum_{k=0}^{n-1} \left| \|Df_{f^k x} e_{f^k x}^s\| - \|Df_{f^k y} e_{f^k y}^s\| \right|.$$

Substituting this into (8.27) and (8.26) and using (8.25), we get

$$(8.28) \quad \begin{aligned} \Delta_n &\leq 2e^{2c_2 n} \sum_{k=0}^{n-1} \left| \|Df_{f^k x} e_{f^k x}^s\| - \|Df_{f^k y} e_{f^k y}^s\| \right| \\ &\leq 2Q_{10} e^{2c_2 n} \sum_{k=0}^{n-1} e^{6\epsilon\gamma\alpha(\ell+k)} d(f^k x, f^k y)^{\alpha\beta}. \end{aligned}$$

Using the bound e^{c_2} for the derivative we get $d(f^k x, f^k y) \leq e^{kc_2} d(x, y)$ and therefore, plugging this into (8.28) and rearranging the terms we get

$$\Delta_n \leq 2Q_{10} e^{2c_2 n} e^{6\epsilon\gamma\alpha\ell} d(x, y)^{\alpha\beta} \sum_{k=0}^{n-1} e^{(6\epsilon\gamma+c_2\beta)\alpha k}.$$

To bound the geometric sum, we write

$$\sum_{k=0}^{n-1} e^{(6\epsilon\gamma+c_2\beta)\alpha k} = \frac{e^{(6\epsilon\gamma+c_2\beta)\alpha n} - 1}{e^{(6\epsilon\gamma+c_2\beta)\alpha} - 1} \leq \frac{e^{(6\epsilon\gamma+c_2\beta)\alpha n}}{e^{(6\epsilon\gamma+c_2\beta)\alpha} - 1}$$

and so we conclude that

$$\Delta_n \leq \frac{2Q_{10}}{e^{(6\epsilon\gamma+c_2\beta)\alpha} - 1} e^{2c_2 n} e^{6\epsilon\gamma\alpha\ell} d(x, y)^{\alpha\beta} e^{(6\epsilon\gamma+c_2\beta)\alpha n}$$

which gives the result. \square

Proof of Proposition 8.8. We want to use (8.23) for large n , and (8.24) for small n ; the transition happens at the point where the two bounds are roughly equal. Thus we choose N such that

$$e^{2\epsilon\ell} e^{-2\chi N} \approx e^{(2+\alpha\beta)c_2 N} e^{6\epsilon\gamma\alpha(\ell+N)} d(x, y)^{\alpha\beta};$$

more precisely, we take

$$(8.29) \quad N = \left\lfloor \frac{2\epsilon\ell - 6\epsilon\gamma\alpha\ell - \alpha\beta \log d(x, y)}{2\chi + (2 + \alpha\beta)c_2 + 6\epsilon\gamma\alpha} \right\rfloor,$$

so that

$$(8.30) \quad \begin{aligned} (2\chi + (2 + \alpha\beta)c_2 + 6\epsilon\gamma\alpha)N &\leq 2\epsilon\ell - 6\epsilon\gamma\alpha\ell - \alpha\beta \log d(x, y) \\ &\leq (2\chi + (2 + \alpha\beta)c_2 + 6\epsilon_0\gamma\alpha)(N + 1). \end{aligned}$$

Note that there is a number $\rho > 0$ which depends only on α, β, ℓ , and ϵ such that the numerator in (8.29) is positive provided $d(x, y) < \rho$. Continuing with this assumption our choice of N and the bound in (8.23) give

$$\begin{aligned} \sum_{n=N}^{\infty} e^{2\lambda n} \Delta_n &\leq \sum_{n=N}^{\infty} 2e^{2\lambda n} e^{2\epsilon\ell} e^{-2\chi n} \\ &\leq \frac{2}{1 - e^{-2(\chi-\lambda)}} e^{2\epsilon\ell} e^{2(-\chi+\lambda)N} = Q_{11} e^{2\epsilon\ell} e^{2(-\chi+\lambda)(N+1)} \end{aligned}$$

where $Q_{11} = 2(1 - e^{-2(\chi-\lambda)})^{-1} e^{2(\chi-\lambda)}$. Then the second inequality in (8.30) and the definitions of the constants ι, η, ζ in (1.6) give

$$(8.31) \quad \begin{aligned} \sum_{n=N}^{\infty} e^{2\lambda n} \Delta_n &\leq Q_{11} e^{2\epsilon\ell} e^{-2(\chi-\lambda) \frac{2\epsilon\ell - 6\epsilon\gamma\alpha\ell - \alpha\beta \log d(x, y)}{6\epsilon_0\gamma\alpha + (2+\alpha\beta)c_2 + 2\chi}} \\ &= Q_{11} e^{2\epsilon\ell} e^{-\iota(2\epsilon\ell - 6\epsilon\gamma\alpha\ell - \alpha\beta \log d(x, y))} \\ &= Q_{11} e^{(2(1-\iota) + 6\gamma\alpha\iota)\epsilon\ell} d(x, y)^{\alpha\beta\iota} \leq Q_{11} e^{\epsilon\eta\ell} d(x, y)^{\zeta}, \end{aligned}$$

where the last inequality uses the fact that $2(1 - \iota) + 6\gamma\alpha\iota \leq 2 + 6\gamma\alpha\iota = \eta$. Turning our attention to the finite part of the sum, (8.24) gives

$$\begin{aligned} \sum_{n=0}^{N-1} e^{2\lambda n} \Delta_n &\leq Q_9 \sum_{n=0}^{N-1} e^{2\lambda n} e^{(2+\alpha\beta)c_2 n} e^{6\epsilon\gamma\alpha(\ell+n)} d(x, y)^{\alpha\beta} \\ &\leq Q_{12} e^{6\epsilon\gamma\alpha\ell} e^{(6\epsilon\gamma\alpha + (2+\alpha\beta)c_2 + 2\lambda)N} d(x, y)^{\alpha\beta} \\ &= Q_{12} e^{6\epsilon\gamma\alpha\ell} e^{(6\epsilon\gamma\alpha + (2+\alpha\beta)c_2 + 2\chi)(1-\iota)N} d(x, y)^{\alpha\beta} \end{aligned}$$

for some constant Q_{12} independent of ℓ, x, y . Applying the first inequality in (8.30) gives

$$e^{6\epsilon\gamma\alpha\ell} e^{(6\epsilon\gamma\alpha + (2+\alpha\beta)c_2 + 2\chi)N} d(x, y)^{\alpha\beta} \leq e^{2\epsilon\ell},$$

and thus

$$\begin{aligned} \sum_{n=0}^{N-1} e^{2\lambda n} \Delta_n &\leq Q_{12} e^{2\epsilon\ell} e^{-\iota(6\epsilon\gamma\alpha + (2+\alpha\beta)c_2 + 2\chi)N} \\ &= Q_{13} e^{2\epsilon\ell} e^{-\iota(6\epsilon\gamma\alpha + (2+\alpha\beta)c_2 + 2\chi)(N+1)} \end{aligned}$$

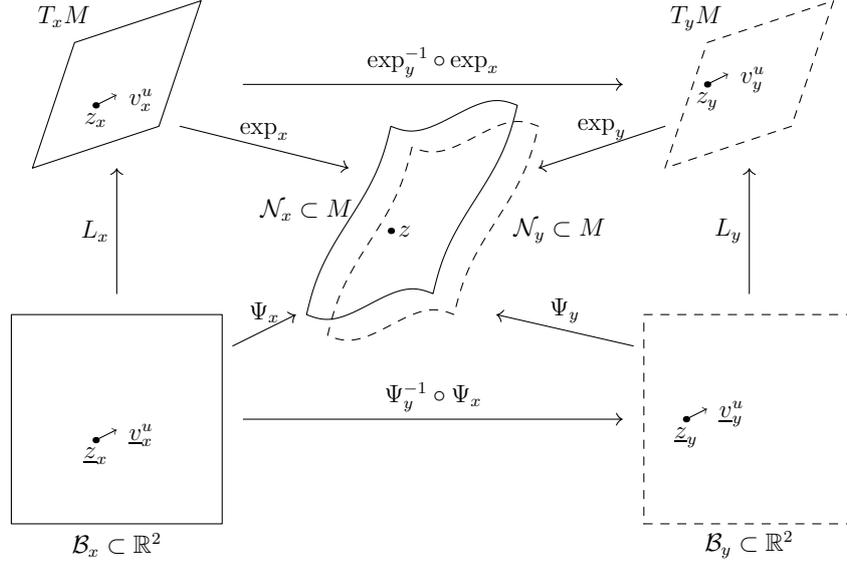


FIGURE 8.1. Applying the transition map.

for $Q_{13} = Q_{12}e^{\iota(6\epsilon\gamma\alpha+(2+\alpha\beta)c_2+2\chi)}$. Now the second inequality in (8.30) gives

$$\begin{aligned} \sum_{n=0}^{N-1} e^{2\lambda n} \Delta_n &\leq Q_{13}e^{2\epsilon\ell} e^{-\iota(2\epsilon\ell-6\epsilon\gamma\alpha-\alpha\beta\log d(x,y))} \\ &= Q_{13}e^{(2(1-\iota)+6\gamma\alpha)\epsilon\ell} d(x,y)^{\alpha\beta\iota} \leq Q_{13}e^{\eta\epsilon\ell} d(x,y)^\zeta, \end{aligned}$$

where the last inequality again uses $2(1-\iota) + 6\gamma\alpha \leq \eta$. Adding (8.31) and using (8.22) completes the proof of Proposition 8.8 in the case $d(x,y) < \rho$. When $d(x,y) \geq \rho$, (8.23) gives

$$(8.32) \quad \Delta_n \leq 2e^{2\epsilon\ell} e^{-2\chi n} \rho^{-\zeta} d(x,y)^\zeta$$

and thus (8.22) gives

$$|s(x) - s(y)| \leq \sum_{n \geq 0} e^{2\lambda n} \Delta_n \leq \frac{2e^{2\epsilon\ell} \rho^{-\zeta}}{1 - e^{-2(x-\lambda)}} d(x,y)^\zeta,$$

which completes the proof because $\eta \geq 1$, so $e^{2\epsilon\ell} \leq e^{2\epsilon\eta\ell}$. \square

8.4. Overlapping derivative estimates. We are now ready to begin the proof of Theorem H. We consider two points $x, y \in \Lambda_\ell$ with the property that $d(x,y) \leq \delta e^{-\lambda\ell}$, as in (8.3), and prove that the corresponding regular neighbourhoods at level ℓ are overlapping, subject to certain conditions on δ . Crucially, these conditions will not depend on ℓ .

In this section we prove the derivative estimates involved in the definition of overlapping charts. In fact, we will prove here a slightly stronger version of (8.1) by showing that (8.1) holds for all $z \in \Psi_x^{-1}(\mathcal{N}_x \cap \mathcal{N}_y)$. The analogous statement (8.2) is completely symmetric.

For $x, y \in \Lambda_\ell$ with $\mathcal{N}_x \cap \mathcal{N}_y \neq \emptyset$, let $z \in \mathcal{N}_x \cap \mathcal{N}_y$ and denote

$$\underline{z}_x := \Psi_x^{-1}(z) \in \mathcal{B}_x, \quad \text{and} \quad \underline{z}_y := \Psi_y^{-1}(z) \in \mathcal{B}_y.$$

Then $(\Psi_y^{-1} \circ \Psi_x)(z_x) = z_y$ and $D_{z_x}(\Psi_y^{-1} \circ \Psi_x): T_{z_x}\mathcal{B}_x \rightarrow T_{z_y}\mathcal{B}_x$. We consider the standard coordinates given by the orthogonal basis (e_1, e_2) in $T_{z_x}\mathcal{B}_x$ and $T_{z_y}\mathcal{B}_x$ and consider an unstable vector

$$\underline{v}_x^u \in \tilde{K}^u \subset T_{z_x}\mathcal{B}_x$$

which we assume is normalized, so that $\|\underline{v}_x^u\| = 1$, and which we write as

$$\underline{v}_x^u = \underline{v}_{1,x}^u e_1 + \underline{v}_{2,x}^u e_2.$$

Then let

$$\underline{v}_y^u := D_{z_x}(\Psi_y^{-1} \circ \Psi_x)(\underline{v}_x^u) = \underline{v}_{1,y}^u e_1 + \underline{v}_{2,y}^u e_2.$$

We will estimate the absolute values of $\underline{v}_{1,y}^u, \underline{v}_{2,y}^u$ in order to prove (8.1). Consider unit vectors $e_x^u, e_x^s \in T_x^M$ and $e_y^u, e_y^s \in T_y^M$, in the directions given by the hyperbolic splitting. Throughout this section, we write $d_{x,y} := d(x, y)$, $u_x = u(x), s_x = s(x)$ to make our computations more compact and easier to read. Observe that

$$\Psi_y^{-1} \circ \Psi_x = L_y^{-1} \circ \exp_y^{-1} \circ \exp_x \circ L_x.$$

Use $\{e_x^u, e_x^s\}$ as a basis for each tangent space to $T_x M$ in the obvious way, and similarly for T_y^M . With respect to these bases and the standard basis $\{e_1, e_2\}$, the derivatives of the maps $L_y^{-1}, \exp_y^{-1} \circ \exp_x$, and L_x are represented by the matrices

$$\begin{pmatrix} u_y & 0 \\ 0 & s_y \end{pmatrix}, \quad \begin{pmatrix} \xi_1^u & \xi_1^s \\ \xi_2^u & \xi_2^s \end{pmatrix}, \quad \begin{pmatrix} u_x^{-1} & 0 \\ 0 & s_x^{-1} \end{pmatrix},$$

respectively, where $\xi_{1/2}^{s/u} \in \mathbb{R}$ are determined by

$$(8.33) \quad \begin{aligned} D_{L_x(z_x)}(\exp_y^{-1} \circ \exp_x)e_x^u &= \xi_1^u e_y^u + \xi_2^u e_y^s, \\ D_{L_x(z_x)}(\exp_y^{-1} \circ \exp_x)e_x^s &= \xi_1^s e_y^u + \xi_2^s e_y^s. \end{aligned}$$

Thus $D_{z_x}(\Psi_y^{-1} \circ \Psi_x)$ has matrix (with respect to $\{e_1, e_2\}$) given by the product of these matrices, which is

$$\begin{pmatrix} u_y & 0 \\ 0 & s_y \end{pmatrix} \begin{pmatrix} u_x^{-1} \xi_1^u & s_x^{-1} \xi_1^s \\ u_x^{-1} \xi_2^u & s_x^{-1} \xi_2^s \end{pmatrix} = \begin{pmatrix} u_y u_x^{-1} \xi_1^u & u_y s_x^{-1} \xi_1^s \\ s_y u_x^{-1} \xi_2^u & s_y s_x^{-1} \xi_2^s \end{pmatrix},$$

and we conclude that

$$(8.34) \quad \underline{v}_{1,y}^u = u_y u_x^{-1} \xi_1^u \underline{v}_{1,x}^u + u_y s_x^{-1} \xi_1^s \underline{v}_{2,x}^u,$$

$$(8.35) \quad \underline{v}_{2,y}^u = s_y u_x^{-1} \xi_2^u \underline{v}_{1,x}^u + s_y s_x^{-1} \xi_2^s \underline{v}_{2,x}^u.$$

We now collect the various estimates which we will plug into these equations to estimate the norms of $\underline{v}_{1,y}^u, \underline{v}_{2,y}^u$.

Lemma 8.11. *There exists a constant $Q_{14} > 0$ such that for every $\ell \in \mathbb{N}$ and $x, y \in \Lambda_\ell$ satisfying (8.3), we have*

$$(8.36) \quad |\underline{v}_{1,x}^u| \geq 1/\sqrt{2} \geq 1/2 \quad \text{and} \quad |\underline{v}_{2,x}^u| \leq e^{-\lambda} \omega |\underline{v}_{1,x}^u|,$$

$$(8.37) \quad |\xi_2^u|, |\xi_1^s| \leq Q_{14} e^{6\epsilon\gamma\ell} d_{x,y}^\beta \quad \text{and} \quad |1 - \xi_1^u|, |1 - \xi_2^s| \leq Q_{14} e^{6\epsilon\gamma\ell} d_{x,y}^\beta,$$

$$(8.38) \quad Q_{14} e^{-\epsilon\ell} \leq u_y s_x^{-1} \leq Q_{14} e^{\epsilon\ell} \quad \text{and} \quad Q_{14} e^{-\epsilon\ell} \leq s_y u_x^{-1} \leq Q_{14} e^{\epsilon\ell},$$

$$(8.39) \quad 1 - Q_{14}e^{(\eta-1)\epsilon\ell}d_{x,y}^\zeta \leq u_y u_x^{-1} \leq 1 + Q_{14}e^{(\eta-1)\epsilon\ell}d_{x,y}^\zeta,$$

$$(8.40) \quad 1 - Q_{14}e^{(\eta-1)\epsilon\ell}d_{x,y}^\zeta \leq s_y s_x^{-1} \leq 1 + Q_{14}e^{(\eta-1)\epsilon\ell}d_{x,y}^\zeta.$$

Proof. Equation (8.36) follows immediately from the fact that \underline{v}_x^u is a unit vector and $\underline{v}_x^u \in \tilde{K}^u$. Equation (8.37) follows from (8.33) and Proposition 8.2 which gives quantitative control on the Hölder dependence of the stable and unstable directions on the base point in Λ_ℓ . Equation (8.38) follows immediately from (6.1). Finally, by (6.1) and Proposition 8.8,

$$\frac{u_y}{u_x} = 1 + \frac{u_y - u_x}{u_x} \geq 1 - \frac{|u_y - u_x|}{u_x} \geq 1 - \frac{Q_8 e^{\epsilon\eta\ell} d_{x,y}^\zeta}{Q_0 \widehat{Q}^{-1} e^{\epsilon\ell}},$$

which gives the first half of (8.39). The upper bound and (8.40) are similar. \square

We are now ready to start estimating the two components $\underline{v}_{1,y}^u, \underline{v}_{2,y}^u$ of \underline{v}_y^u . We estimate each one separately. Once we have proved these lemmas, we will be in a position to give the conditions on $\delta > 0$, which we stress will be independent of ℓ .

Lemma 8.12. *For every $\ell \in \mathbb{N}$ and $x, y \in \Lambda_\ell$ satisfying (8.3), we have*

$$|\underline{v}_{1,y}^u| \geq (1 - Q_{14}\delta^\beta)(1 - Q_{14}\delta^\zeta)|\underline{v}_{1,x}^u| - Q_{14}^2\delta^\beta.$$

Proof. Using (8.37) and (8.39), we have the following estimate for the first term of (8.34):

$$|u_y u_x^{-1} \underline{v}_{1,x}^u \xi_1^u| \geq (1 - Q_{14}e^{6\epsilon\gamma\ell}d_{x,y}^\beta)(1 - Q_{14}e^{(\eta-1)\epsilon\ell}d_{x,y}^\zeta)|\underline{v}_{1,x}^u|.$$

Now (8.3) and the bounds on ϵ in (1.7) give

$$(8.41) \quad \begin{aligned} e^{6\epsilon\gamma\ell}d_{x,y}^\beta &\leq e^{6\epsilon\gamma\ell}\delta^\beta e^{-\beta\lambda\ell} = \delta^\beta e^{(6\epsilon\gamma-\beta\lambda)\ell} \leq \delta^\beta, \\ e^{(\eta-1)\epsilon\ell}d_{x,y}^\zeta &\leq e^{(\eta-1)\epsilon\ell}\delta^\zeta e^{-\zeta\lambda\ell} = \delta^\zeta e^{((\eta-1)\epsilon-\zeta\lambda)\ell} \leq \delta^\zeta, \end{aligned}$$

and thus

$$(8.42) \quad |u_y u_x^{-1} \xi_1^u \underline{v}_{1,x}^u| \geq (1 - Q_{14}\delta^\beta)(1 - Q_{14}\delta^\zeta)|\underline{v}_{1,x}^u|.$$

For the second term of (8.34), by (8.37) and (8.38), and using $|\underline{v}_{2,x}^u| \leq 1$, we have

$$|u_y s_x^{-1} \xi_1^s \underline{v}_{2,x}^u| \leq Q_{14}^2 e^{\epsilon\ell} e^{6\epsilon\gamma\ell} d_{x,y}^\beta \leq Q_{14}^2 e^{7\epsilon\gamma\ell} d_{x,y}^\beta,$$

where the second inequality uses the fact that $\gamma \geq 1$. As in (8.41) above,

$$(8.43) \quad e^{7\epsilon\gamma\ell}d_{x,y}^\beta \leq \delta^\beta e^{(7\epsilon\gamma-\beta\lambda)\ell} \leq \delta^\beta,$$

and thus $|u_y s_x^{-1} \xi_1^s \underline{v}_{2,x}^u| \leq Q_{14}^2 \delta^\beta$. Subtracting this from (8.42) and recalling (8.34) proves the lemma. \square

Lemma 8.13. *For every $\ell \in \mathbb{N}$ and $x, y \in \Lambda_\ell$ satisfying (8.3), we have*

$$|\underline{v}_{2,y}^u| \leq (1 + Q_{14}\delta^\beta)(1 + Q_{14}\delta^\zeta)|\underline{v}_{2,x}^u| + Q_{14}^2\delta^\beta.$$

Proof. Computations nearly identical to those in the previous lemma let us bound the first term in (8.35) as follows:

$$\begin{aligned} |s_y s_x^{-1} \xi_2^s \underline{v}_{2,x}^u| &\leq (1 + Q_{14} e^{(\eta-1)\epsilon\ell} d_{x,y}^\zeta) (1 + Q_{14} e^{6\epsilon\gamma\ell} d_{x,y}^\beta) |\underline{v}_{2,x}^u| \\ &\leq (1 + Q_{14} \delta^\zeta) (1 + Q_{14} \delta^\beta) |\underline{v}_{2,x}^u|. \end{aligned}$$

Similarly computations for the second term of (8.35) give

$$|s_y u_x^{-1} \xi_2^u \underline{v}_{1,x}^u| \leq Q_{14}^2 e^{\epsilon\ell} e^{6\epsilon\gamma\ell} d_{x,y}^\beta \leq Q_{14}^2 \delta^\beta.$$

Adding these estimates together proves the lemma. \square

Proof of Theorem H (derivative estimates). We can now prove the first part of Theorem H concerning properties (8.1) and (8.2) in the definition of overlapping charts. Start by requiring that $\delta > 0$ is sufficiently small that $Q_{14} \delta^\beta < 1$ and $Q_{14} \delta^\zeta < 1$. (Further conditions will come later.) Lemmas 8.12 and 8.13 give

$$\begin{aligned} \frac{|\underline{v}_{2,y}^u|}{|\underline{v}_{1,y}^u|} &\leq \frac{(1 + Q_{14} \delta^\beta) (1 + Q_{14} \delta^\zeta) |\underline{v}_{2,x}^u| + Q_{14}^2 \delta^\beta}{(1 - Q_{14} \delta^\beta) (1 - Q_{14} \delta^\zeta) |\underline{v}_{1,x}^u| - Q_{14}^2 \delta^\beta} \\ &\leq \frac{(1 + Q_{14} \delta^\beta) (1 + Q_{14} \delta^\zeta) e^{-\lambda\omega} |\underline{v}_{1,x}^u| + Q_{14}^2 \delta^\beta}{(1 - Q_{14} \delta^\beta) (1 - Q_{14} \delta^\zeta) |\underline{v}_{1,x}^u| - Q_{14}^2 \delta^\beta}, \end{aligned}$$

where the second inequality uses the fact that $\underline{v}_x^u \in \tilde{K}^u$. Note that the function $t \mapsto \frac{at+b}{ct+d}$ is decreasing in t when $ad-bc < 0$, which is the case for the expression above, and thus we can obtain an upper bound by observing that (8.36) gives $|\underline{v}_{1,x}^u| \geq \frac{1}{2}$, so monotonicity gives

$$(8.44) \quad \frac{|\underline{v}_{2,y}^u|}{|\underline{v}_{1,y}^u|} \leq \frac{(1 + Q_{14} \delta^\beta) (1 + Q_{14} \delta^\zeta) e^{-\lambda\omega/2} + Q_{14}^2 \delta^\beta}{(1 - Q_{14} \delta^\beta) (1 - Q_{14} \delta^\zeta) / 2 - Q_{14}^2 \delta^\beta}.$$

For sufficiently small $\delta > 0$, the right-hand side is $< \omega$, which implies that $\underline{v}_y^u \in K^u$ as required by the first part of (8.1).

For the second part of (8.1), we observe that

$$\frac{\|\underline{v}_y^u\|}{\|\underline{v}_x^u\|} \geq \frac{|\underline{v}_{1,y}^u|}{\sqrt{|\underline{v}_{1,x}^u|^2 + |\underline{v}_{2,x}^u|^2}} \geq \frac{((1 - Q_{14} \delta^\beta) (1 - Q_{14} \delta^\zeta) - Q_{14}^2 \delta^\beta) |\underline{v}_{1,x}^u|}{\sqrt{1 + e^{-2\lambda\omega^2}} |\underline{v}_{1,x}^u|},$$

where the second inequality uses Lemma 8.12 and (8.36) for the numerator and the fact that $\underline{v}_x^u \in \tilde{K}^u$ for the denominator. Recall from (5.9) that $1/\sqrt{1 + e^{-2\lambda\omega^2}} > e^{-\lambda/24}$; thus we can choose δ small enough that the right-hand side of the above expression is $> e^{-\lambda/24}$, which proves the second half of (8.1). Condition (8.2) follows by analogous arguments. \square

8.5. Overlapping stable and unstable strips. We now complete the proof of Theorem H by showing that if $\gamma \subset \tilde{\mathcal{N}}_{x,\ell}^s$ is a full length strongly stable curve, then it completely crosses $\tilde{\mathcal{N}}_{y,\ell}^u$. The other three required conditions obtained by interchanging x/y and stable/unstable are proved analogously.

We start with a couple of simple Lemmas relating the distance $d(x, y)$ between two points and the amount of overlap of their regular neighbourhoods.

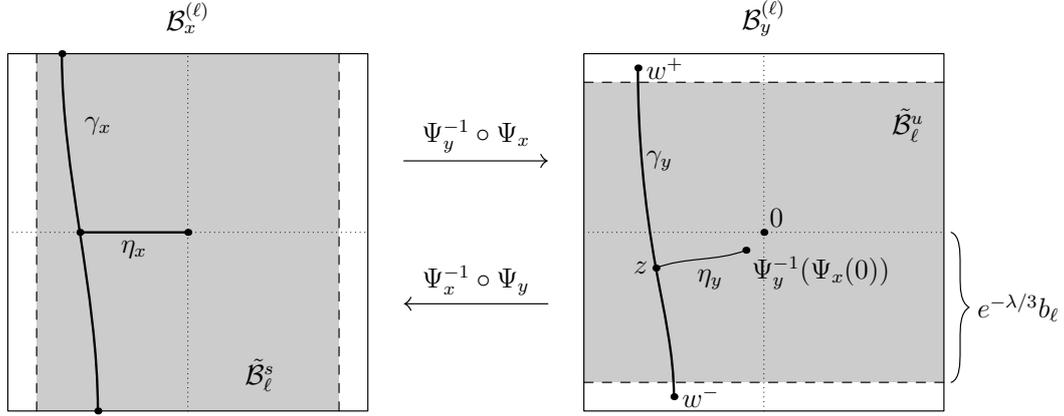


FIGURE 8.2. Proving Theorem H.

Lemma 8.14. *There exists a constant $Q_{15} > 0$ such that for every $\ell \geq 1$ and every $x, y \in \Lambda_\ell$ satisfying $d(x, y) \leq Q_{15}e^{-2\ell}b_\ell$, we have $x, y \in \mathcal{N}_x^{(\ell)} \cap \mathcal{N}_y^{(\ell)}$.*

Proof. It is enough to prove that for every $\ell \geq 1$ and $x \in \Lambda_\ell$, the regular neighbourhood $\mathcal{N}_x^{(\ell)} := \Psi_x(\mathcal{B}_x^{(\ell)}) = \exp_x(L_x(\mathcal{B}_x^{(\ell)}))$ contains a ball centred at x of radius $\gtrsim e^{-2\ell}b_\ell$, where \gtrsim means that we have \geq up to a multiplicative constant that is independent of x, ℓ . To see that this holds, consider first the map $L_x: \mathcal{B}_x^{(\ell)} \rightarrow T_xM$. By (5.2) this maps $\mathcal{B}_x^{(\ell)}$ to a parallelogram with sides parallel to the stable and unstable directions E_x^s, E_x^u and which, by (5.2) and (6.1), have length $\approx u(x)^{-1}b_\ell \gtrsim e^{-\ell}b_\ell$ and $\approx s(x)^{-1}b_\ell \gtrsim e^{-\ell}b_\ell$ respectively. By condition **(H2)** and the definition of Λ_ℓ in (1.2) we have $\angle(E_x^s, E_x^u) \geq e^{-\ell}$ and therefore the result follows in T_xM by elementary trigonometry. Taking Q_{15} sufficiently small, the map \exp_x is arbitrarily close to an isometry and thus the result follows also for the regular neighborhoods on the manifold. \square

Lemma 8.15. *There exists $Q_{16} > 0$ such that for $\ell \geq 1$ and $x, y \in \Lambda_\ell$,*

$$\text{if } y \in \mathcal{N}_x^{(\ell)} \quad \text{then} \quad \|\Psi_x^{-1} \circ \Psi_y(0)\| \leq Q_{16}e^{2\ell}d(x, y).$$

Proof. If $y \in \mathcal{N}_x^{(\ell)}$ then $\Psi_x^{-1} \circ \Psi_y(0) = \Psi_x^{-1}(y)$ is well defined. Therefore $\|\Psi_x^{-1} \circ \Psi_y(0)\| = \|\Psi_x^{-1}(y)\| = \|\Psi_x^{-1}(y) - 0\| = \|\Psi_x^{-1}(y) - \Psi_x^{-1}(x)\|$ and so we just need to estimate the Lipschitz constant of Ψ_x^{-1} . By definition we have $\Psi_x^{-1} = L_x^{-1} \circ \exp_x^{-1}$ and the result follows using (6.6) in Lemma 6.4 and the fact that \exp_x^{-1} is close to an isometry. \square

Proof of Theorem H (stable and unstable strips). We can now complete the proof of Theorem H. Let $x, y \in \Lambda_\ell$ with $d(x, y) \leq \delta e^{-\lambda\ell}$ as in (8.3). Then since $b_\ell = Ce^{-2\ell/\alpha}$ for some constant $C > 0$, we have

$$(8.45) \quad d(x, y)e^{2\ell}b_\ell^{-1} \leq \delta e^{-\lambda\ell} e^{2\ell} C^{-1} e^{2\ell/\alpha} = \delta C^{-1} e^{(2\ell(1+\frac{1}{\alpha})-\lambda)\ell} \leq \delta C^{-1},$$

where the last inequality uses (1.7). By making δ sufficiently small that $\delta C^{-1} \leq Q_{15}$, we guarantee that the hypothesis of Lemma 8.14 is satisfied.

Now let γ be a full length strongly stable curve in $\tilde{\mathcal{N}}_{x,\ell}^s$, and consider the curves $\gamma_x = \Psi_x^{-1}(\gamma) \subset \tilde{\mathcal{B}}_\ell^s$ and $\gamma_y = \Psi_y^{-1}(\gamma) \subset \mathcal{B}_y^{(\ell)} \subset \mathcal{B}_y^{(\ell)}$, as shown in Figure 8.2. Let η_x be the segment of the x -axis in $\mathcal{B}_x^{(\ell)}$ that connects γ_x to 0. Let $\eta_y = \Psi_y^{-1} \circ \Psi_x(\eta_x)$. Let $z \in \mathcal{B}_y^{(\ell)}$ be the intersection point of η_y and γ_y , and let w^\pm be the endpoints of γ_y . Writing $z = z_1e_1 + z_2e_2$ and similarly for w^\pm , our goal is to show that $|w_2^\pm| \geq e^{-\lambda/3}b_\ell$.

We give the proof for w^+ ; the proof for w^- is similar. By Lemma 8.15 and (8.45), the point $v := \Psi_y^{-1}(x) = \Psi_y^{-1}(\Psi_x(0))$ has

$$(8.46) \quad \|v\| \leq Q_{16}e^{2\ell}d(x, y) \leq Q_{16}C^{-1}\delta b_\ell.$$

By (8.1), η_y is an unstable curve connecting z and v with length $\leq e^{\lambda/24}e^{-\lambda/3}b_\ell$ (using the fact that η_x has length at most $e^{-\lambda/3}b_\ell$). Since γ_y is a stable curve, we have

$$|w_1^+ - z_1| \leq 2\omega b_\ell,$$

and thus

$$|w_1^+| \leq |v_1| + |z_1 - v_1| + |w_1^+ - z_1| \leq Q_{16}C^{-1}\delta b_\ell + e^{\lambda/24}e^{-\lambda/3}b_\ell + 2\omega b_\ell.$$

By (5.9) we can choose $\delta > 0$ small enough that

$$(8.47) \quad Q_{16}C^{-1}\delta + e^{\lambda/24}e^{-\lambda/3} + 2\omega < 1,$$

and we conclude that $|w_1^+| < b_\ell$, so w^+ is not on a vertical boundary of $\mathcal{B}_y^{(\ell)}$.

If w^+ is on the top boundary of $\mathcal{B}_y^{(\ell)}$, then there is nothing to prove, so we can assume that the part of γ_y running from z to w^+ is the image of the top half of γ_x under the transition map. By (8.1), this part of γ_y is a stable curve with length $\geq e^{-\lambda/24}b_\ell$ (using the fact that the top half of γ_x has length at least b_ℓ). Since the length of this part of γ_y is at most $\sqrt{1 + \omega^2}|w_2^+ - z_2|$, we conclude that

$$(8.48) \quad |w_2^+ - z_2| \geq \frac{e^{-\lambda/24}b_\ell}{\sqrt{1 + \omega^2}}.$$

As argued above, η_y has length $\leq e^{\lambda/24}b_\ell$; on the other hand since it is a stable curve, its length is at least

$$\sqrt{|z_1 - v_1|^2 + |z_2 - v_2|^2} \geq \sqrt{\omega^{-2} + 1}|z_2 - v_2|,$$

and we conclude that

$$(8.49) \quad |z_2 - v_2| \leq \frac{\omega e^{\lambda/24}b_\ell}{\sqrt{1 + \omega^2}}.$$

Combining (8.46), (8.48), and (8.49) gives

$$\begin{aligned} |w_2^+| &\geq |w_2^+ - z_2| - |z_2 - v_2| - |v_2| \geq \frac{e^{-\lambda/24}b_\ell}{\sqrt{1 + \omega^2}} - \frac{\omega e^{\lambda/24}b_\ell}{\sqrt{1 + \omega^2}} - Q_{16}\delta b_\ell \\ &= \left(\frac{e^{-\lambda/24} - \omega e^{\lambda/24}}{\sqrt{1 + \omega^2}} - Q_{16}\delta \right) b_\ell \geq (e^{-\lambda/4} - Q_{16}\delta) b_\ell, \end{aligned}$$

where the last inequality uses (5.9). As long as δ is small enough that

$$(8.50) \quad e^{-\lambda/4} - Q_{16}\delta > e^{-\lambda/3},$$

this gives $|w_2^+| \geq e^{-\lambda/3}b_\ell$, completing the proof. \square

9. PSEUDO-ORBITS, BRANCHES, SHADOWING: PROOFS OF THEOREMS E AND F

We are now ready to prove Theorems E and F.

9.1. Regular branches: Proof of Theorem E. The two fundamental ingredients in the proof of Theorem E are Theorem H and Theorem G, which is essentially the special case of Theorem E where $k = 1$ and the pseudo-orbit is in fact a real orbit.

Fix $\delta > 0$ sufficiently small so that the conclusions of Theorem H hold. To prove the first part of Theorem E, we observe that if $\bar{x} = (x_0, \dots, x_k)$ is an $(\bar{\ell}, \delta, \lambda)$ -pseudo-orbit, then by Theorem H, the Lyapunov charts $\mathcal{N}_{x_j}^{(\ell_j)}$ and $\mathcal{N}_{f(x_{j-1})}^{(\ell_j)}$ are overlapping for every $1 \leq j \leq k$. By Theorem G, for each $0 \leq j < k$, the sets

$$\begin{aligned} \mathcal{B}_{x_j, (\ell_j, \ell_{j+1})}^{s,1} &= \Psi_{x_j}^{-1}(\mathcal{N}_{x_j}^{(\ell_j)} \cap f^{-1}\mathcal{N}_{f(x_j)}^{\ell_{j+1}}), \\ \mathcal{B}_{f(x_j), (\ell_j, \ell_{j+1})}^{u,1} &= \Psi_{f(x_j)}^{-1}(f(\mathcal{N}_{x_j}^{(\ell_j)}) \cap \mathcal{N}_{f(x_j)}^{(\ell_{j+1})}) \end{aligned}$$

from (7.1) are strongly stable and strongly unstable strips in $\tilde{\mathcal{B}}_{b_{\ell_j}}^s$ and $\tilde{\mathcal{B}}_{b_{\ell_{j+1}}}^u$, respectively, and f_{x_j} is a diffeomorphism between them that satisfies the inclusions and estimates in (7.3). Since the Lyapunov charts $\mathcal{N}_{x_{j+1}}^{(\ell_{j+1})}$ and $\mathcal{N}_{f(x_j)}^{(\ell_{j+1})}$ are overlapping, we conclude that $\Psi_{x_{j+1}}^{-1} \circ \Psi_{f(x_j)}(\mathcal{B}_{f(x_j), (\ell_j, \ell_{j+1})}^{u,1})$ is an unstable strip in $\mathcal{B}_{x_{j+1}}^{(\ell_{j+1})}$, and thus its preimage under $\Psi_{x_{j+1}}^{-1} \circ f \circ \Psi_{x_j}$ is a stable strip in $\mathcal{B}_{x_j}^{(\ell_j)}$. Thus we have

$$\mathcal{B}_{x_0}^{(\ell_0)} \xrightarrow{\Psi_{x_0}^{-1} \circ f \circ \Psi_{x_1}} \mathcal{B}_{x_1}^{(\ell_1)} \xrightarrow{\Psi_{x_1}^{-1} \circ f \circ \Psi_{x_2}} \mathcal{B}_{x_2}^{(\ell_2)} \rightarrow \dots \rightarrow \mathcal{B}_{x_{k-1}}^{(\ell_{k-1})} \xrightarrow{\Psi_{x_{k-1}}^{-1} \circ f \circ \Psi_{x_k}} \mathcal{B}_{x_k}^{(\ell_k)}$$

where the maps are not defined on the entirety of the indicated domain, but only on a stable strip, and the corresponding image is an unstable strip. In particular, taking the composition of all the maps we see that (5.12) with $j = 0$ defines a stable strip $\mathcal{B}_{\bar{x}}^0 \subset \mathcal{B}_{x_0}^{(\ell_0)}$ that is mapped to an unstable strip $\mathcal{B}_{\bar{x}} \subset \mathcal{B}_{x_k}^{(\ell_k)}$ by $\Psi_{x_k}^{-1} \circ f^k \circ \Psi_{x_0}$. This proves the first property in the definition of an $\bar{\ell}$ -regular branch. For the second, we observe that by (7.3), each $f_{x_{j-1}} = \Psi_{f(x_{j-1})}^{-1} \circ f \circ \Psi_{x_{j-1}}$ has a derivative that maps K^u into \tilde{K}^u , and that the transition map $\Psi_{x_j}^{-1} \circ \Psi_{f(x_{j-1})}$ maps this into K^u by the definition of overlapping charts; moreover, the first map above expands each vector in K^u by a factor of at least $e^{\lambda/2}$, and so after composing with the transition map, the derivative of $\Psi_{x_j}^{-1} \circ f \circ \Psi_{x_{j-1}}$ expands each vector in K^u by a factor of at least $e^{\lambda/2}e^{-\lambda/24}$. Iterating completes the proof of Theorem E.

9.2. Shadowing: Proof of Theorem F. We prove Theorem F using Theorem E and the definition of regular branch in Definition 5.5, together with the hyperbolicity estimates from (5.16). Start by choosing the constants as in the assumption of Theorem F.

First we prove that $\bigcap_{n \in \mathbb{Z}} f^{-n} \Psi_{x_n}(\mathcal{B}_{x_n}^{(\ell_n)})$ is a single point, which gives existence and uniqueness of the shadowing point y . Given $n \in \mathbb{N}$, let $\bar{\ell}^{n,+} := (\ell_0, \dots, \ell_n)$ and $\bar{x}^{n,+} := (x_0, \dots, x_n)$, so that $\bar{x}^{n,+}$ is an $(\bar{\ell}^{n,+}, \delta, \lambda)$ -pseudo-orbit, which by Theorem E determines an $\bar{\ell}^{n,+}$ -regular branch. Let $\mathcal{B}_{\bar{x}}^0(n) \subset \mathcal{B}_{x_0}^{(\ell_0)}$ be the corresponding stable strip. The expansion estimates in Definition 5.5 imply that any unstable curve crossing this strip has length $\leq 2b_0 e^{-\lambda n/3}$. Similarly, let $\mathcal{B}_{\bar{x}}^0(-n) \subset \mathcal{B}_{x_0}^{(\ell_0)}$ be the unstable strip corresponding to the regular branch for the pseudo-orbit (x_{-n}, \dots, x_0) , and note that any stable curve crossing this strip has length $\leq 2b_0 e^{-\lambda n/3}$. It follows that $\mathcal{B}_{\bar{x}}^0(n) \cap \mathcal{B}_{\bar{x}}^0(-n)$ has diameter $\leq 4b_0 e^{-\lambda n/3}$, and thus $\bigcap_{n \in \mathbb{N}} \mathcal{B}_{\bar{x}}^0(n) \cap \mathcal{B}_{\bar{x}}^0(-n)$ is a single point \underline{y} (since the sets are nested and compact). The image $\Psi_{x_0}(\underline{y})$ is the unique shadowing point y .

Now we establish regularity of y . By Definition 5.5, the sequence of sets $K_n^s(\underline{y}) := (D_{\underline{y}} f_{\bar{x}}^{0,n})^{-1}(K^s) \subset T_{\underline{y}} \mathcal{B}_{x_0}^{(\ell_0)} = \mathbb{R}^2$ is nested, and $\bigcap_{n \in \mathbb{N}} K_n^s(\underline{y}) =: E_{\underline{y}}^s \subset \mathbb{R}^2$ is a one-dimensional subspace. Let $E_y^s := D_{\underline{y}} \Psi_{x_0}(E_{\underline{y}}^s)$, and define E_y^u similarly using $K_n^u(\underline{y}) := (D_{\underline{y}} f_{\bar{x}}^{0,-n})^{-1}(K^u)$. It follows immediately from the definitions that $D_y f E_y^{u/s} = E_{f(y)}^{u/s}$, so it remains to show that this invariant splitting satisfies **(H1)**–**(H3)**.

To this end, let $e_y^{u/s} \in E_y^{u/s}$ be unit vectors, and let $v_n^{u/s} := D_y f^n(e_y^{u/s})$ for all $n \in \mathbb{Z}$. We prove the estimates in the first line of **(H3)**; the ones in the second line are similar. Observe that near y , we have $f^n = \Psi_{x_n} \circ f_{\bar{x}}^{0,n} \circ \Psi_{x_0}^{-1}$, so

$$\begin{aligned} \|v_n^s\| &\leq \|D_{f_{\bar{x}}^{0,n}(\underline{y})} \Psi_{x_n}\| \cdot \|D_{\underline{y}} f_{\bar{x}}^{0,n}\| \cdot \|D_{\underline{y}} \Psi_{x_0}^{-1}\| \\ &\leq 2 \cdot e^{-\lambda n/3} \cdot 4Q_0 \widehat{Q}^{-1} e^{2\epsilon \ell_0} = \widehat{Q}^{-1} e^{2\epsilon \ell_0} e^{-\lambda n/3} \leq \widehat{Q}^{-1} e^{2\epsilon \ell_0} e^{-\lambda n/4}, \end{aligned}$$

where the second inequality uses Definition 5.5 for the second bound and (5.15) for the first and third. A similar computation using $f_{\bar{x}}^{n,0}$ gives

$$1 = \|e_n^u\| \leq \widehat{Q}^{-1} e^{2\epsilon \ell_n} e^{-\lambda n/3} \|v_n^u\|,$$

and thus

$$\|v_n^u\| \geq \widehat{Q} e^{-2\epsilon \ell_n} e^{\lambda n/3} \geq \widehat{Q} e^{-2\epsilon \ell_0} e^{(-2\epsilon + \lambda/3)n} \geq \widehat{Q} e^{-2\epsilon \ell_0} e^{\lambda n/4}.$$

This establishes **(H3)** with $C(y) = \widehat{Q}^{-1} e^{2\epsilon \ell_0}$. To prove **(H1)** for C , observe that $f(y)$ is the shadowing orbit for $\sigma \bar{x}$ with regularity sequence $\bar{\ell}$, so that $C(f(y)) = \widehat{Q}^{-1} e^{2\epsilon \ell_1}$, and thus

$$(9.1) \quad e^{-2\epsilon} \leq C(f(y))/C(y) \leq e^{2\epsilon}.$$

Now we need to estimate $\angle(E_y^s, E_y^u)$. First observe that $\angle(E_y^s, E_y^u) \geq \|e_y^s - e_y^u\|$ since the angle represents the length of the arc of the unit circle joining the endpoints of e_y^s and e_y^u , while the right-hand side is the length of the straight

line joining them. Let $\underline{v}^{u/s} := D_y \Psi_{x_0}^{-1} e_y^{u/s}$ and observe that $\|\underline{v}^{u/s}\| \geq \frac{1}{2}$ by the first estimate in (5.15). Moreover, $\underline{v}^{u/s} \in K^{s/u}$, so the endpoint of \underline{v}^u lies in the region of \mathbb{R}^2 given by

$$\{(x, y) \in \mathbb{R}^2 : |y| \leq \omega|x| \text{ and } x^2 + y^2 \geq 1/4\},$$

while the endpoint of \underline{v}^u lies in the region

$$\{(x, y) \in \mathbb{R}^2 : |x| \leq \omega|y| \text{ and } x^2 + y^2 \geq 1/4\},$$

Thus, by (5.9), $\|\underline{v}^u - \underline{v}^s\| \geq 1/2$,¹³ and we conclude that

$$\frac{1}{2} \leq \|\underline{v}^u - \underline{v}^s\| \leq \|D_y \Psi_{x_0}^{-1}\| \|e_y^u - e_y^s\| \leq 4Q_0 \widehat{Q}^{-1} e^{2\epsilon\ell_0} \angle(E_y^s, E_y^u)$$

using the second inequality in (5.15). Thus

$$\angle(E_y^s, E_y^u) \geq \widehat{Q} e^{-2\epsilon\ell_0},$$

so putting $K(y) := \widehat{Q} e^{-2\epsilon\ell_0}$ establishes **(H2)**, and **(H1)** for K follows just as it did for C . We conclude that the set Λ' of shadowing points is $(\lambda/4, 2\epsilon)$ -hyperbolic. Moreover, to find which Λ'_ℓ contains y , we write

$$C(y) = \widehat{Q}^{-1} e^{2\epsilon\ell_0} \leq e^{2\epsilon\ell}$$

and find that this holds as soon as $\ell \geq \ell_0 - \frac{1}{2\epsilon} \log \widehat{Q}$. A similar computation with $K(y)$ shows that y is $(\lambda/4, 2\epsilon, \ell_0 - \lfloor \frac{1}{2\epsilon} \log \widehat{Q} \rfloor)$ -regular.

Part III. Nice Rectangles and Young Towers

In this third and final part of the paper, we apply the general results stated in Theorem E and F, and proved in Part II above, to our particular setting in order to prove Theorems B, C and D. As mentioned above, Theorem A follows directly from Theorems B and C, and therefore this completes the proofs of all our results. The sections are organized as follows: in §10 we prove Theorem D, which is essentially a reformulation of Theorem E in the setting of almost returns to nice domains, in §11 we show that Theorem D implies Theorem B, and in §12 we prove Theorem C.

10. HYPERBOLIC BRANCHES IN NICE DOMAINS: PROOF OF THEOREM D

The proof of Theorem D consists of two parts. First we show that every almost return gives rise to a pseudo-orbit and thus, by Theorem E, to a regular branch, which satisfies the hyperbolicity estimates given in (5.16). Then we show that this regular branch can be “restricted” to give a hyperbolic branch in the nice domain Γ_{pq} . This second part of the proof does not explicitly require Theorem E, it only uses the existence of a regular branch, but does use in an essential way the fact that Γ_{pq} is a nice domain.

¹³An elementary computation shows that the optimal lower bound is $(1-\omega)/\sqrt{2(1+\omega^2)}$.

To begin, let $C_\ell \geq 1$ be the constant given in Theorem 1.11, c_2 as given in (1.5), and $\delta > 0$ as in Theorem E. Then we let

$$(10.1) \quad r := \frac{\delta e^{-\lambda\ell} e^{-c_2}}{2C_\ell} \leq \delta.$$

For generality we state the following Lemma for almost returns in a slightly more general setting than that of Theorem D, without any explicit references to rectangles or nice domains.

Lemma 10.1. *If $x, y \in \Lambda_\ell$ and $k \geq 1$ are such that $f^k(V_x^s) \cap V_y^u \neq \emptyset$, and $z \in f^k(V_x^s) \cap V_y^u$ satisfies $d(z, y) < r$ and $d(f^{-k}(z), x) < r$, then the sequence $\bar{x} = (x_0, \dots, x_k)$ given by $x_j = f^j(x)$ for $0 \leq j \leq k/2$ and $x_j = f^{j-k}(y)$ for $k/2 < j \leq k$ is an $(\bar{\ell}, \delta, \lambda)$ -pseudo-orbit for $\ell_j = \min(\ell + j, \ell + k - j)$.*

Proof. Write $i = \lfloor k/2 \rfloor$. By assumption $z \in V_y^u$ and $f^{-k}(z) \in V_x^s$ and therefore by the assumptions of the Lemma and Theorem 1.11 we have

$$d(f^i(x), f^{i-k}(z)) \leq C_\ell e^{-\lambda i} d(x, f^{-k}(z)) \leq C_\ell e^{-\lambda i} r \leq \frac{1}{2} \delta e^{-\lambda(\ell+i)},$$

and

$$d(f^{i-k}(z), f^{i-k}(y)) \leq C_\ell e^{-\lambda(k-i)} d(z, y) \leq C_\ell e^{-\lambda(k-i)} r \leq \frac{1}{2} \delta e^{-\lambda(\ell+i)},$$

and thus $d(f(x_i), x_{i+1}) = d(f^i(x), f^{i-k}(y)) \leq \delta e^{-\lambda(\ell+i)}$. Since $f(x_j) = x_{j+1}$ for all $j \neq i$, this completes the proof. \square

Consider now the setting of Theorem D: suppose Γ is a nice set with $\text{diam}(\Gamma_{pq}) < r$ and suppose $x \in \Gamma$ has an almost return to Γ at time $k \in T\mathbb{N}$. Then the assumptions of Lemma 10.1 are satisfied and there is an $(\bar{\ell}, \delta, \lambda)$ -pseudo-orbit $\bar{x} = (x_0, \dots, x_k)$ starting and ending inside Γ_{pq} . Moreover, notice that $d(p, x_1) = d(p, f(x_0)) \leq e^{c_2} d(p, x_0) \leq \delta \leq e^{c_2} \text{diam}(\Gamma_{pq}) \leq e^{c_2} r < \delta e^{-\lambda\ell}$ and, by a similar calculation, $d(x_{k-1}, p) < \delta$, and therefore the sequence

$$\bar{p} := (p, x_1, \dots, x_{k-1}, p)$$

is also an $(\bar{\ell}, \delta, \lambda)$ -pseudo-orbit. Considering the Lyapunov chart $\Psi_p: \mathcal{B}_p^{(\ell)} \rightarrow \mathcal{N}_p^{(\ell)}$, by Theorem E there is an ℓ -regular branch from $\mathcal{B}_p^{(\ell)}$ to itself associated to this pseudo-orbit, and we have the corresponding maps

$$(10.2) \quad f_{\bar{p}}^{0,k}: \mathcal{B}_{\bar{p}}^0 \rightarrow \mathcal{B}_{\bar{p}}^k \quad \text{and} \quad f^k: \mathcal{N}_{\bar{p}}^0 \rightarrow \mathcal{N}_{\bar{p}}^k$$

at the level of Lyapunov charts and of the manifold respectively, recall (5.13). For this branch we have the hyperbolicity estimates given in (5.16), which show that it is a $(\widehat{Q}^{-1} e^{2\ell}, \lambda/3)$ -hyperbolic branch. Moreover, concatenating any finite sequence of such branches gives a new ℓ -regular branch that is associated to the concatenated pseudo-orbit, and thus has the same hyperbolicity estimates given by (5.16). Thus the collection of such branches satisfies the concatenation property.

Remark 10.2. We emphasize that these are not yet the hyperbolic branches we require for Γ_{pq} as in Definition 4.4. Indeed, these branches are constructed on the scale of the Lyapunov chart which *a priori* may be significantly bigger than the scale of the nice domain Γ_{pq} . The strips $\mathcal{N}_{\bar{p}}^0, \mathcal{N}_{\bar{p}}^k$ intersect Γ_{pq} but may extend across the boundary of Γ_{pq} . We therefore need to “restrict” these branches to Γ_{pq} and produce Γ_{pq} -strips $\widehat{C}^s \subset \mathcal{N}_{\bar{p}}^0$ and $\widehat{C}^u \subset \mathcal{N}_{\bar{p}}^k$ such that f^k maps \widehat{C}^s onto \widehat{C}^u . Since these are subsets of the larger strips $\mathcal{N}_{\bar{p}}^0, \mathcal{N}_{\bar{p}}^k$ and the cones $K_{p,y}^{s/u} \subset T_y M$ defined in (5.11) give cone-fields over Γ_{pq} that are adapted to the set Γ , the restricted strips will automatically inherit the hyperbolicity and concatenation properties.

The remaining part of the argument is essentially topological, and this is where the niceness assumption plays a crucial role. Indeed, the crucial consequence of niceness is formalized in the following statement.

Lemma 10.3. *Let Γ be a nice set and suppose that some $x \in \Gamma$ has an almost return to Γ at a time $k \in \mathbb{N}T$. Then $f^k(W_x^s) \subset \Gamma_{pq}$.*

Proof. Suppose by contradiction that the conclusion does not hold. Then $f^k(W_x^s)$ must intersect one of $W_{p/q}^u$, but this implies that the image under f^{-k} of this intersection point lies in the interior of Γ_{pq} , which is forbidden by niceness. \square

Let

$$\gamma_p^u := W_p^u \cap \mathcal{N}_{\bar{x},p}^0 \quad \text{and} \quad \gamma_q^u := W_q^u \cap \mathcal{N}_{\bar{x},p}^0.$$

Lemma 10.4. *γ_p^u, γ_q^u are strictly contained in W_p^u, W_q^u respectively.*

Proof. We prove the statement for γ_p^u , the same argument applies to γ_q^u . Suppose by contradiction that $\gamma_p^u = W_p^u$. This implies $\gamma_p^u \subset \mathcal{N}_{\bar{x},p}^0$ and, since by assumption k is a multiple of T and therefore both p, q are fixed points for f^k , it also implies that $W_p^s, W_q^s \subset \mathcal{N}_{\bar{x},p}^0$ and therefore in particular that $p, q \in \mathcal{N}_{\bar{x},p}^0$. But this is not possible because a regular branch cannot contain more than one fixed point for f^k . \square

Lemma 10.5. *$f^k(\gamma_p^u), f^k(\gamma_q^u)$ are full length unstable curves in Γ_{pq} .*

Proof. We prove the statement for γ_p^u , the same argument applies to γ_q^u . There are essentially two possible “configurations” depending on whether one or none of the endpoints of γ_p^u coincide with the endpoints of W_p^u (the case where both endpoints coincide with endpoints of W_p^u is excluded by Lemma 10.4). If neither endpoint coincides with the endpoints of W_p^u (as illustrated in Figure 10.1) then both endpoints lie on the stable boundaries of $\mathcal{N}_{\bar{x},p}^0$ and therefore their images lie on the stable boundaries of $\mathcal{N}_{\bar{x},p}^k$. By Lemma 10.3, $f^k(\gamma_p^u)$ intersects Γ_{pq} and therefore is necessarily full length in Γ_{pq} . If one of the endpoints coincides with an endpoint of W_p^u then this endpoint lies on either W_q^s or W_p^s and, since both p and q are fixed by f^k , the image of this endpoint necessarily lies on W_q^s or W_p^s (in the latter case it is in fact equal to

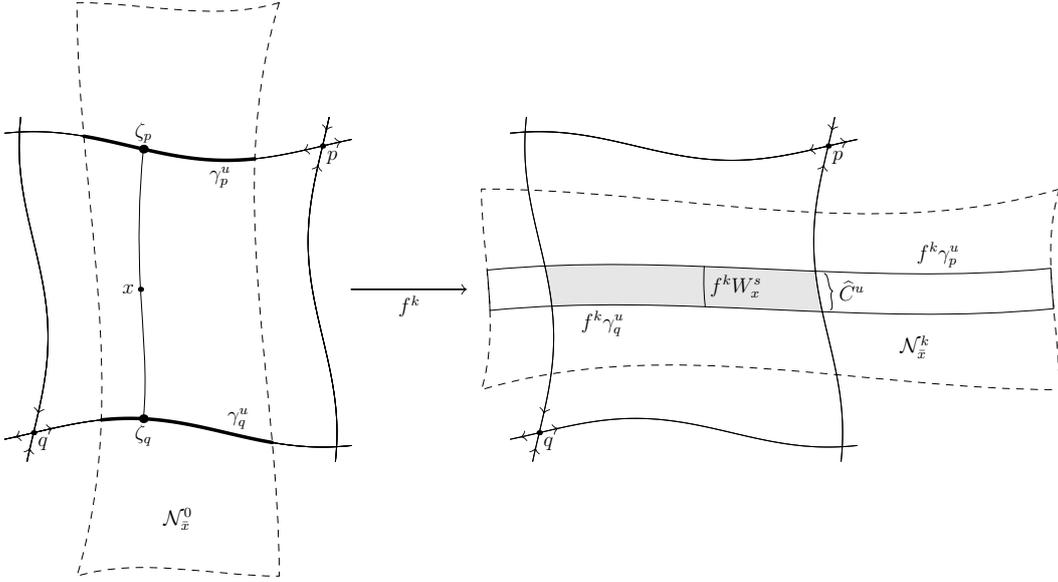


FIGURE 10.1. Proving Theorem D.

p). Since the other endpoint lies, as in the first case, on the stable boundary of $\mathcal{N}_{\bar{x},p}^0$, its image lies on the stable boundary of $\mathcal{N}_{\bar{x},p}^k$ and therefore $f^k(\gamma_p^u)$ is necessarily full length in Γ_{pq} . \square

Proof of Theorem D. By Lemma 10.5, $f^k(\gamma_p^u), f^k(\gamma_q^u)$ are full length unstable curves in Γ_{pq} and therefore they define an unstable strip \widehat{C}^u in Γ_{pq} , whose preimage $\widehat{C}^s := f^{-k}(\widehat{C}^u)$ is a stable strip in Γ_{pq} , thus yielding the desired hyperbolic branch. The required hyperbolicity estimates are inherited from the regular branch of which this hyperbolic branch is a subset. \square

11. BUILDING A TOWER OUT OF HYPERBOLIC BRANCHES: PROOF OF THEOREM B

In this section we prove Theorem B. In §11.1 we introduce some definitions and notation and reduce the proof of Theorem B to three Propositions 11.3, 11.4, and 11.5. We then prove each Proposition in its own subsection.

11.1. Saturation and Young towers. The first step in proving Theorem B is to consider the collection of all hyperbolic branches which arise as a consequence of almost returns to a nice recurrent set A . This collection is non-empty because the set A satisfies the T -return property (by assumption) and the hyperbolic branch property (by Theorem D). Notice moreover that since f is a diffeomorphism of a compact manifold the derivative of f is bounded and so for each $i \geq 1$ there can be at most a finite number κ_i of hyperbolic branches of order i . We can index them by a set of the form

$$I := \{ij : i \in \mathbb{N}_T, j \in \{1, \dots, \kappa_i\}\},$$

where i gives the almost return time, which is a multiple of T by Definition 2.3, and j indexes the κ_i hyperbolic branches with return time i . We therefore

let

$$(11.1) \quad \mathfrak{C} = \mathfrak{C}(A) := \{f^i: \widehat{C}_{ij}^s \rightarrow \widehat{C}_{ij}^u\}_{ij \in I}.$$

denote the collection of hyperbolic branches associated to almost returns to A . We write \mathfrak{C}^* for the collection of all branches obtained by concatenating finitely many elements of \mathfrak{C} .

Remark 11.1. Note that elements of \mathfrak{C}^* do not necessarily correspond to almost returns of A ; indeed, \mathfrak{C}^* may contain branches $f^k: \widehat{C}^s \rightarrow \widehat{C}^u$ such that \widehat{C}^s and \widehat{C}^u are disjoint from A . This can occur if two branches $f^i: \widehat{C}_1^s \rightarrow \widehat{C}_1^u$ and $f^j: \widehat{C}_2^s \rightarrow \widehat{C}_2^u$ generated by almost returns have a concatenated branch $f^{i+j}: \widehat{C}^s \rightarrow \widehat{C}^u$ (recall Figure 4.2) with the property that the part of A in \widehat{C}_1^s lies entirely outside of \widehat{C}^s , and similarly for \widehat{C}_2^u .

Definition 11.2 (Saturated Rectangle). A rectangle $\Gamma \subset \Gamma_{pq} \cap \Lambda_\ell$ is *saturated* (for \mathfrak{C}) if for all $ij \in I$,

$$(11.2) \quad C_{ij}^s := f^{-i}(\widehat{C}_{ij}^u \cap C^s) \subset C^s \quad \text{and} \quad C_{ij}^u := f^i(\widehat{C}_{ij}^s \cap C^u) \subset C^u,$$

where $C^{s/u} = \bigcup_{x \in \Gamma} W_x^{s/u}$ as in (2.1). In this case, it immediately follows from iterating (11.2) that Γ is saturated for \mathfrak{C}^* as well.

Theorem B is an immediate consequence of the following three propositions (recall Definitions 4.6 and 4.7 for almost returns and the hyperbolic branch property).

Proposition 11.3. *Let A be a nice almost recurrent (χ, ϵ, ℓ) -regular set with the (C, λ) -hyperbolic branch property and let \mathfrak{C} denote the corresponding collection of hyperbolic branches. Then there exists a nice recurrent rectangle Γ satisfying $A \subseteq \Gamma \subseteq \Gamma_{pq}$ with the (C, λ) -hyperbolic branch property, which has \mathfrak{C}^* as its collection of hyperbolic branches, and is saturated for \mathfrak{C}^* . The rectangle Γ is $(\lambda/4, 2\epsilon, \ell + \ell')$ -regular, with $\ell' = \lceil \frac{1}{2\epsilon} \log \widehat{Q} \rceil$.*

Proposition 11.4. *Let $\Gamma \subset \Gamma_{pq}$ be a nice recurrent saturated rectangle satisfying the (C, λ) -hyperbolic branch property. Then Γ supports a First T -return Topological Young Tower.*

Proposition 11.5. *Let $\Gamma \subset \Gamma_{pq}$ be a nice fat strongly recurrent saturated rectangle satisfying the (C, λ) -hyperbolic branch property. Then Γ supports a First T -return Young Tower with integrable return times.*

Remark 11.6. The rectangle Γ constructed in Proposition 11.3 has the following ‘‘dynamical local product structure’’ property: if $x, y \in \Gamma$ and $i \in \mathbb{Z}$ are such that $[x, y, i] \neq \emptyset$, then $[x, y, i]$ is a single point that belongs to Γ . In particular, the branch f^i corresponding to the almost return gives a genuine return of this point back to Γ , so every branch corresponding to an almost return actually contains a true return.

We will prove each of the Propositions above in its own subsection.

11.2. Saturated rectangles. In this section we prove Proposition 11.3. We will define Γ as the “maximal invariant set” for the dynamics induced by the hyperbolic branches. Indeed, the setting can be thought of as a generalization of the standard horseshoe where we define a maximal invariant set as the points which remain in the strips for all forward and backward iterations. The key difference is that in the horseshoe setting we have just two (or more generally a finite number of) branches with pairwise disjoint stable strips and pairwise disjoint unstable strips, all of which have the same return time, whereas in our setting a point x may belong to many, generally infinitely many, stable strips with varying return times. We therefore need to make a choice as to which of these branches we use as this may affect whether the image of x belongs to one of the existing stable strips, and thus whether x belongs to such a maximal invariant set. We formalize this construction as follows.

Definition 11.7 (Hyperbolic sequences). A sequence $\mathfrak{h}^+ = \{i_m j_m\}_{m=0}^\infty$ with $i_m j_m \in I$ is a *forward hyperbolic sequence* for $x \in \Gamma_{pq}$ if for all $m \geq 0$ we have

$$(11.3) \quad f^{i_0+i_1+\dots+i_{m-1}}(x) \in \widehat{C}_{i_m j_m}^s.$$

Similarly, $\mathfrak{h}^- = \{i_m j_m\}_{m=-\infty}^{-1}$ is a *backward hyperbolic sequence* for $x \in \Gamma_{pq}$ if for all $m < 0$ we have

$$(11.4) \quad f^{-(i_m+\dots+i_{-1})}(x) \in \widehat{C}_{i_m j_m}^s.$$

If \mathfrak{h}^- and \mathfrak{h}^+ are backward and forward hyperbolic sequences for x , their concatenation $\mathfrak{h} = \{i_m j_m\}_{m \in \mathbb{Z}}$ is called a *hyperbolic sequence* for x .

A point x may or may not admit a forward or backward hyperbolic sequence and, if it does, these sequences need not be uniquely defined. Consider the sets

$$\begin{aligned} C^s &:= \{x \in \Gamma_{pq} \mid x \text{ has a forward hyperbolic sequence } \mathfrak{h}^+\}, \\ C^u &:= \{x \in \Gamma_{pq} \mid x \text{ has a backward hyperbolic sequence } \mathfrak{h}^-\}. \end{aligned}$$

Notice that we have used here the same notation used in (2.1) to describe the structure of rectangles. This is intentional; it is motivated and justified by the following statement.

Lemma 11.8. *The following set is a rectangle:*

$$(11.5) \quad \Gamma := C^s \cap C^u = \{x \in \Gamma_{pq} \mid x \text{ has a hyperbolic sequence } \mathfrak{h}\}.$$

We think of Γ as the “maximal invariant set” of \mathfrak{C} .

Proof of Lemma 11.8. The fact that Γ is a rectangle follows by essentially the same arguments as in the standard horseshoe setting. Let $\mathfrak{h} = \{i_m j_m\}_{m \in \mathbb{Z}}$ denote a hyperbolic sequence for the point x , and \mathfrak{h}^\pm its forward and backward parts. Then the set of points which admit a finite piece $\{i_0 j_0, i_1 j_1, \dots, i_m j_m\}$ of this hyperbolic sequence is a full length stable strip whose width tends to zero exponentially fast in m and therefore the intersection of all such strips is

precisely W_x^s , the local stable curve of x restricted to Γ_{pq} , which therefore in particular is full length and can be characterized as

$$(11.6) \quad W_x^s = \{y \in \Gamma_{pq} : \mathfrak{h}^+ \text{ is a forward hyperbolic sequence for } y\}.$$

A completely analogous argument shows that every $x \in \Gamma$ has a full length local unstable curve

$$(11.7) \quad W_x^u = \{y \in \Gamma_{pq} : \mathfrak{h}^- \text{ is a backward hyperbolic sequence for } y\}.$$

This implies that for any $x, y \in \Gamma$ the intersection $W_x^s \cap W_y^u$ consists of a single point z . Moreover, x has a forward hyperbolic sequence \mathfrak{h}_x^+ and y has a backward hyperbolic sequence \mathfrak{h}_y^- ; writing \mathfrak{h} for the concatenation of these two sequences, it follows from (11.6) and (11.7) that \mathfrak{h} is a hyperbolic sequence for z , so $z \in \Gamma$.

The fact that Γ is $(\lambda/4, 2\epsilon, \ell + \ell')$ -regular for $\ell' = \lceil |\frac{1}{2\epsilon} \log \widehat{Q}| \rceil$ follows from Theorem F. \square

In order to prove that Γ has the hyperbolic branch property, we need the following result about hyperbolic branches.

Lemma 11.9. *Let Γ_{pq} be a nice domain. Then for any hyperbolic branch $f^i: \widehat{C}^s \rightarrow \widehat{C}^u$ we have $\text{Int}(f^k(\widehat{C}^s)) \cap \partial\Gamma_{pq} = \emptyset$ for all $k = 0, \dots, i$ that are multiples of T . Moreover, if $f^{i'}: \widehat{C}'^s \rightarrow \widehat{C}'^u$ is any other hyperbolic branch, then the corresponding stable (resp. unstable) strips are either nested or disjoint.*

Proof. The first statement is automatic for $k = 0, i$. Suppose that there exists some $k \in \{1, \dots, i-1\}$ such that $\text{Int}(f^k(\widehat{C}^s)) \cap \partial\Gamma_{pq} \neq \emptyset$. Then we must have $\text{Int}(f^k(\widehat{C}^s)) \cap (W_q^s \cup W_p^s) \neq \emptyset$ or $\text{Int}(f^k(\widehat{C}^s)) \cap (W_q^u \cup W_p^u) \neq \emptyset$ (or both). In the first case, iterating forward by $i-k$ iterates, this would imply that $\text{Int}(\widehat{C}^u) \cap f^{i-k}(W_q^s \cup W_p^s) \neq \emptyset$, contradicting the niceness property of Γ_{pq} . Similarly, in the second case, iterating backwards by k iterates, this would imply $\text{Int}(\widehat{C}^s) \cap f^{-k}(W_q^u \cup W_p^u) \neq \emptyset$, contradicting niceness.

For the second statement, assume without loss of generality that $i \leq i'$. Suppose by contradiction that the two stable strips $\widehat{C}^s, \widehat{C}'^s \neq \emptyset$ are neither nested nor disjoint (the argument for unstable strips is exactly the same). The stable boundaries of $\widehat{C}^s, \widehat{C}'^s$ are pieces of the global stable curves of p, q and therefore cannot intersect and so the intersection $\widehat{C}^s \cap \widehat{C}'^s$ is a non-empty stable strip. Thus each stable strip has one of the components of its stable boundary inside the interior of the other stable strip, see first figure in Fig. 11.1. It follows that $f^i(\widehat{C}'^s)$ contains a piece of the stable boundary of Γ_{pq} in its interior, contradicting the first statement proved above. \square

Lemma 11.10. *The rectangle Γ defined in (11.5) satisfies the following properties.*

- (1) Γ is recurrent.
- (2) Every T -return produces a hyperbolic branch: if $x, f^i(x) \in \Gamma$ for some $i \in \mathbb{N}$ that is a multiple of T , then there is a hyperbolic branch $f^i: \widehat{C}^s \rightarrow \widehat{C}^u$ such that $x \in \widehat{C}^s$.

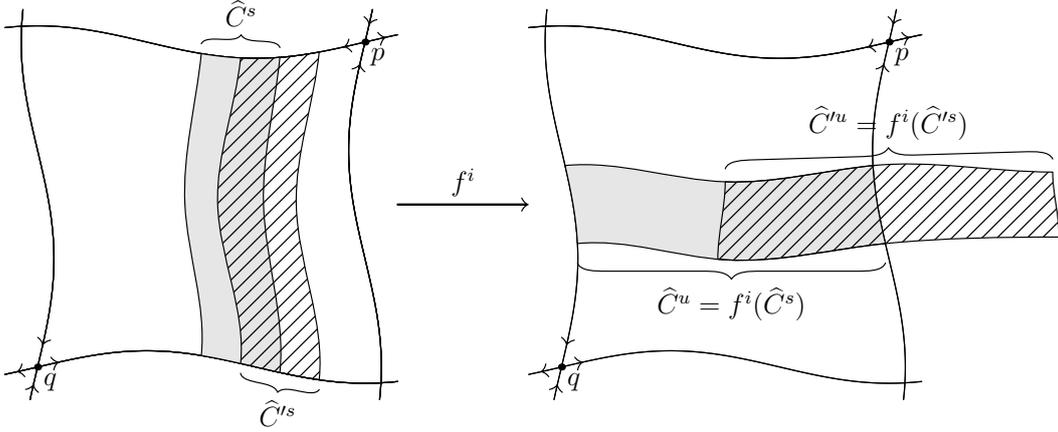


FIGURE 11.1. Two overlapping branches leads to a contradiction (Lemma 11.9).

(3) *The set of all hyperbolic branches produced in this way is \mathfrak{C}^* .*

Proof. The fact that Γ is recurrent follows almost immediately from the definition; if $\mathfrak{h} = \{i_m j_m\}_{m \in \mathbb{Z}}$ is a hyperbolic sequence for $x \in \Gamma$, then $f^{i_0}(x)$ and $f^{-i_{-1}}(x)$ have hyperbolic sequences given by shifting \mathfrak{h} one index in either direction, hence i_0 and $-i_{-1}$ are return times to Γ .

Now we prove that every T -return gives a hyperbolic branch. Let $z \in \Gamma$ and $\tau \in T\mathbb{N}$ be such that $z' := f^\tau z \in \Gamma$. We can suppose without loss of generality that τ is the first return time of z to Γ since if we prove the hyperbolic branch property for first return times it follows for any return time just by composing the branches; recall that part of the hyperbolic branch property is that the concatenation of a sequence of (C, λ) -hyperbolic branches is itself a (C, λ) -hyperbolic branch.

Let $\mathfrak{h}_z = \{i_m j_m\}$ and $\mathfrak{h}_{z'} = \{i'_m j'_m\}$ be hyperbolic sequences for z and z' , chosen so that i_0 and i'_{-1} are minimal over all such sequences. Note that in particular this implies that $\tau \leq \min\{i_0, i'_{-1}\}$. By the definition of hyperbolic sequence we have

$$z \in \widehat{C}_{i_0 j_0}^s \cap \widehat{C}_{i_{-1} j_{-1}}^u \quad \text{and} \quad z' \in \widehat{C}_{i'_0 j'_0}^s \cap \widehat{C}_{i'_{-1} j'_{-1}}^u;$$

in particular, $z \in \widehat{C}_{i_0 j_0}^s$ and $z' \in \widehat{C}_{i'_{-1} j'_{-1}}^u$.

Sublemma 11.11. *There exist $x \in A \cap \widehat{C}_{i_0 j_0}^s$ and $y' \in A \cap \widehat{C}_{i'_{-1} j'_{-1}}^u$ such that $f^\tau(W_x^s) \cap W_{y'}^u \neq \emptyset$.*

Proof of Sublemma 11.11. Notice first of all that by assumption $z \in \widehat{C}_{i_0 j_0}^s$ and $f^\tau(z) = z' \in (\widehat{C}_{i'_{-1} j'_{-1}}^u)$; thus $z \in \widehat{C}_{i_0 j_0}^s \cap f^{-\tau}(\widehat{C}_{i'_{-1} j'_{-1}}^u)$ and we conclude that

$$(11.8) \quad \widehat{C}_{i_0 j_0}^s \cap f^{-\tau}(\widehat{C}_{i'_{-1} j'_{-1}}^u) \neq \emptyset.$$

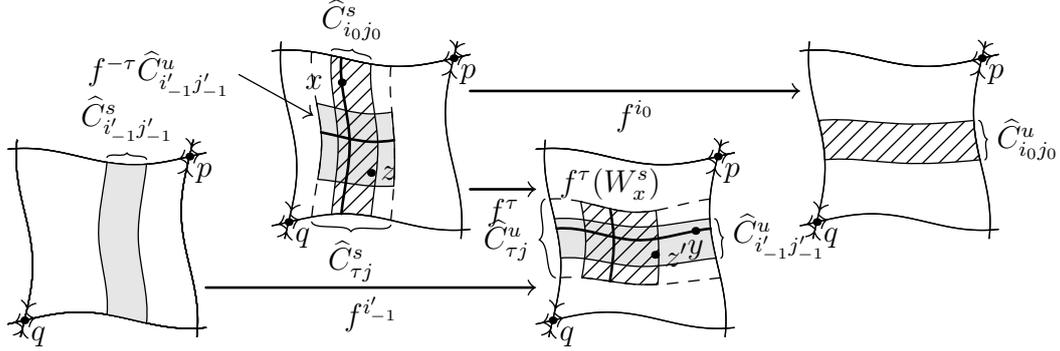


FIGURE 11.2. Proving Lemma 11.10.

In particular this implies $f^{-\tau}(\widehat{C}_{i'_{-1}j'_{-1}}^u) \cap \Gamma_{pq} \neq \emptyset$ and, since $\tau \leq i'_{-1}$, by Lemma 11.9 we have $f^{-\tau}(\widehat{C}_{i'_{-1}j'_{-1}}^u) \cap \partial\Gamma_{pq} = \emptyset$ from which it follows that

$$(11.9) \quad f^{-\tau}(\widehat{C}_{i'_{-1}j'_{-1}}^u) \subset \Gamma_{pq}.$$

Now we claim that

$$(11.10) \quad f^{-\tau}(\partial^s \widehat{C}_{i'_{-1}j'_{-1}}^u) \cap \text{Int}(\widehat{C}_{i_0j_0}^s) = \emptyset,$$

as shown in Figure 11.2. To see this, observe that if the intersection in (11.10) is non-empty, then iterating by f^τ gives $(W_p^s \cup W_q^s) \cap f^\tau(\text{Int}(\widehat{C}_{i_0j_0}^s)) \neq \emptyset$ since $\partial^s \widehat{C}_{i'_{-1}j'_{-1}}^u \subset W_p^s \cup W_q^s$. Iterating again by $f^{i_0-\tau}$ gives $f^{i_0-\tau}(W_p^s \cup W_q^s) \cap \text{Int} \widehat{C}_{i_0j_0}^u \neq \emptyset$, which contradicts the niceness property of Γ_{pq} since $\widehat{C}_{i_0j_0}^u \subset \Gamma_{pq}$. Thus (11.10) holds, and together with (11.8) and (11.9) this implies that $f^{-\tau}(\widehat{C}_{i'_{-1}j'_{-1}}^u)$ fully crosses $\widehat{C}_{i_0j_0}^s$ in the unstable direction, as shown in Figure 11.2. Now we can choose any $x \in A \cap \widehat{C}_{i_0j_0}^s$ and $y' \in A \cap \widehat{C}_{i'_{-1}j'_{-1}}^u$ and get $f^{-\tau}(W_{y'}^u) \cap W_x^s \neq \emptyset$; this implies $W_{y'}^u \cap f^\tau W_x^s \neq \emptyset$ as required. \square

Returning to the proof of Lemma 11.10, by Sublemma 11.11 there exists $x \in A$ that has an almost return to A at time τ . Since almost returns of A give hyperbolic branches, we see that the collection \mathfrak{C} contains a hyperbolic branch $f^\tau: \widehat{C}_{\tau j}^s \rightarrow \widehat{C}_{\tau j}^u$ for which $x \in \widehat{C}_{\tau j}^s$ and $y' \in \widehat{C}_{\tau j}^u$.

It follows that $\widehat{C}_{\tau j}^s$ intersects $\widehat{C}_{i_0j_0}^s$; by Lemma 11.9 these two stable strips must be nested, and since $\tau \leq i_0$ (being a first return) we conclude that $z \in \widehat{C}_{i_0j_0}^s \subset \widehat{C}_{\tau j}^s$. Similarly, $\widehat{C}_{\tau j}^u$ intersects $\widehat{C}_{i'_{-1}j'_{-1}}^u$, and the same argument shows that $\widehat{C}_{\tau j}^u \supset \widehat{C}_{i'_{-1}j'_{-1}}^u \ni z'$. Thus $f^\tau: \widehat{C}_{\tau j}^s \rightarrow \widehat{C}_{\tau j}^u$ is the branch whose existence we were to prove; this completes the proof of Lemma 11.10. \square

Lemma 11.12. Γ is saturated.

Proof. Given $w \in f^{-i}(\widehat{C}_{ij}^u \cap C^s)$ we have $f^i(w) \in C^s$ and therefore $f^i(w)$ has a forward hyperbolic sequence $\{i_m j_m\}_{m \geq 0}$. Therefore w also has a forward hyperbolic sequence obtained by letting ij be the first term of the sequence and

$\{i_{m-1}j_{m-1}\}_{m \in \mathbb{N}}$ as the remaining terms and therefore $w \in C^s$. A completely analogous argument works for the unstable leaves to show that Γ is saturated. \square

11.3. Topological Young Towers. In this section we prove Proposition 11.4. Let $\Gamma = C^s \cap C^u$ be a nice T-recurrent *saturated* rectangle satisfying the (C, λ) -hyperbolic branch property and let \mathfrak{C} denote the collection of hyperbolic branches associated to returns to Γ . To each such branch $f^i: \widehat{C}_{ij}^s \rightarrow \widehat{C}_{ij}^u$ we will associate an s -subset $\Gamma_{ij}^s \subset \widehat{C}_{ij}^s \cap \Gamma$ and $\Gamma_{ij}^u \subset \widehat{C}_{ij}^u \cap \Gamma$ such that $f^i: \Gamma_{ij}^s \rightarrow \Gamma_{ij}^u$ is a bijection. We stress that both inclusions are in general proper; roughly speaking, the reason for this is that there may be some $x \in \widehat{C}_{ij}^s \cap \Gamma$ for which $f^i(x) \notin \Gamma$, and such points must be excluded from Γ_{ij}^s ; similarly for $x \in \widehat{C}_{ij}^u$ and $f^{-i}(x)$. To define $\Gamma_{ij}^{s,u}$, first recall from (11.2) that for $ij \in I$ we write

$$(11.11) \quad \begin{aligned} C_{ij}^s &= f^{-i}(\widehat{C}_{ij}^u \cap C^s) = \widehat{C}_{ij}^s \cap f^{-i}(C^s), \\ C_{ij}^u &= f^i(\widehat{C}_{ij}^s \cap C^u) = \widehat{C}_{ij}^u \cap f^i(C^u); \end{aligned}$$

then let

$$(11.12) \quad \Gamma_{ij}^s := C_{ij}^s \cap C^u, \quad \Gamma_{ij}^u := C_{ij}^u \cap C^s.$$

Notice that C_{ij}^s, C_{ij}^u are collections of stable and unstable leaves respectively, whereas $\Gamma_{ij}^s, \Gamma_{ij}^u$ may be Cantor sets.

Lemma 11.13. *For every $ij \in I$, $\Gamma_{ij}^s, \Gamma_{ij}^u$ are s -subsets and u -subsets respectively of Γ and $f^i(\Gamma_{ij}^s) = \Gamma_{ij}^u$. Moreover, if $x \in \Gamma$ and $i \in \mathbb{N}$ are such that $f^i(x) \in \Gamma$, then $x \in \Gamma_{ij}^s$ for some $ij \in I$; in particular this implies that $\Gamma = \bigcup_{ij \in I} \Gamma_{ij}^s = \bigcup_{ij \in I} \Gamma_{ij}^u$.*

Proof. By the saturation assumption, $C_{ij}^s \subseteq C^s, C_{ij}^u \subseteq C^u$ and therefore $\Gamma_{ij}^s := C_{ij}^s \cap C^u \subseteq C^s \cap C^u = \Gamma$ and $\Gamma_{ij}^u := C_{ij}^u \cap C^s \subseteq C^u \cap C^s = \Gamma$ and so $\Gamma_{ij}^s, \Gamma_{ij}^u \subseteq \Gamma$. Since C_{ij}^s is a union of stable leaves and C_{ij}^u is a union of unstable leaves, the sets $\Gamma_{ij}^s := C_{ij}^s \cap C^u$ and $\Gamma_{ij}^u := C_{ij}^u \cap C^s$ are s -subsets and u -subsets, respectively, of Γ . Moreover, directly from the definitions we have

$$f^i(\Gamma_{ij}^s) = f^i(\widehat{C}_{ij}^s) \cap C^s \cap f^i(C^u) = f^i(\widehat{C}_{ij}^s \cap C^u) \cap C^s = \Gamma_{ij}^u.$$

For the second statement, let $x \in \Gamma$ and $i \in \mathbb{N}$ be such that $f^i(x) \in \Gamma$. Since Γ is a hyperbolically recurrent rectangle, there is a hyperbolic branch $f^i: \widehat{C}_{ij}^s \rightarrow \widehat{C}_{ij}^u$ in \mathfrak{C} such that $x \in \widehat{C}_{ij}^s \cap \Gamma$ and $f^i(x) \in \widehat{C}_{ij}^u \cap \Gamma$. Since $\Gamma \subset C^s$, the definition of C_{ij}^s gives

$$x = f^{-i}(f^i(x)) \in f^{-i}(\widehat{C}_{ij}^u \cap \Gamma) \subset f^{-i}(\widehat{C}_{ij}^u \cap C^s) = C_{ij}^s.$$

Since we also have $x \in \Gamma = C^s \cap C^u \subseteq C^u$ it follows that $x \in C_{ij}^s \cap C^u = \Gamma_{ij}^s$. The final assertion follows because Γ is recurrent and so every $x \in \Gamma$ has some $i, i' \in \mathbb{N}$ such that $f^i(x), f^{-i'}(x) \in \Gamma$. \square

Lemma 11.13 shows that there exists a *cover* of Γ by s -subsets and another by u -subsets satisfying the required Markov property. It does not however claim that this cover is a *partition* of Γ as required by the definition, i.e. that

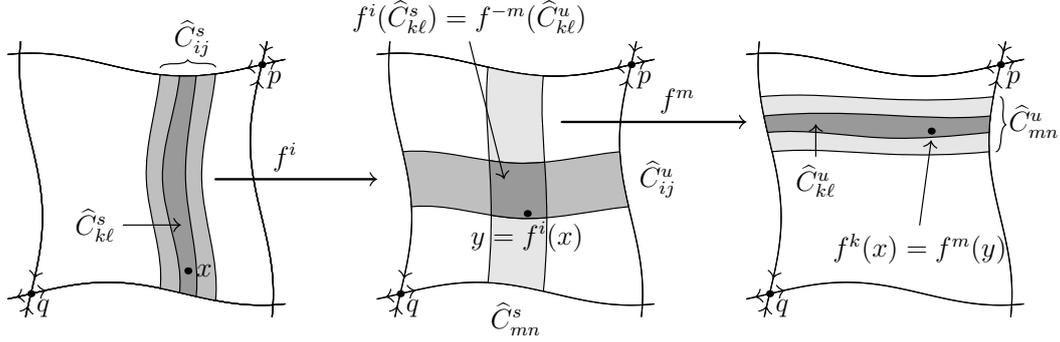


FIGURE 11.3. Proof of Sublemma 11.16.

the s -subsets of the cover are disjoint. In fact, it is not generally the case that they are disjoint, but we can pass to a subcover with pairwise disjoint elements by using Lemma 11.9.

Lemma 11.14. *Let $kl \in I$ and suppose there exists $x \in \Gamma_{kl}^s$ and $0 < i < k$ such that $f^i(x) \in \Gamma$. Then there exist $ij \in I$ such that $\Gamma_{kl}^s \subseteq \Gamma_{ij}^s$. In particular all $\{\Gamma_{ij}^s\}_{ij \in I}$ are pairwise either nested or disjoint.*

We will prove this lemma momentarily.

Remark 11.15. Notice that the last statement in Lemma 11.14 does not follow directly from Lemma 11.9. Indeed, the fact that two stable strips $\widehat{C}_{ij}^s, \widehat{C}_{kl}^s$ are nested does not *a priori* imply that the corresponding sets C_{ij}^s, C_{kl}^s are either disjoint or nested, recall (11.2), and therefore also does not *a priori* imply that $\Gamma_{ij}^s, \Gamma_{kl}^s$ are either disjoint or nested, recall (11.12).

Sublemma 11.16. *Letting $m = k - i$, there exist $ij, mn \in I$ such that $\widehat{C}_{kl}^s \subseteq \widehat{C}_{ij}^s, \widehat{C}_{kl}^u \subseteq \widehat{C}_{mn}^u$, and such that $f^i(\widehat{C}_{kl}^s) = \widehat{C}_{ij}^u \cap \widehat{C}_{mn}^s = f^{-m}(\widehat{C}_{kl}^u)$.*

Proof. From Lemma 11.13 we have $x \in \Gamma_{ij}^s$ for some $ij \in I$. Therefore $x \in \Gamma_{ij}^s \cap \Gamma_{kl}^s$ and so $x \in \widehat{C}_{ij}^s \cap \widehat{C}_{kl}^s$ and in particular $\widehat{C}_{ij}^s \cap \widehat{C}_{kl}^s \neq \emptyset$ and therefore, by Lemma 11.9, $\widehat{C}_{kl}^s \subseteq \widehat{C}_{ij}^s$, as shown in Figure 11.3. Also, from Lemma 11.13, $x \in \Gamma_{kl}^s$ implies $x \in \Gamma$ and $f^k(x) \in \Gamma$ and therefore, letting $y = f^i(x) \in \Gamma$ we have $f^m(y) = f^m(f^i(x)) = f^k(x) \in \Gamma$. Thus there exists $mn \in I$ such that $y \in \widehat{C}_{mn}^s$. We therefore have $f^m(y) \in f^m(\widehat{C}_{mn}^s) = \widehat{C}_{mn}^u$ and also $f^m(y) = f^k(x) \in \Gamma_{kl}^u \subseteq \widehat{C}_{kl}^u$, and therefore $\widehat{C}_{mn}^u \cap \widehat{C}_{kl}^u \neq \emptyset$ and thus, since $m < k$, $\widehat{C}_{kl}^u \subseteq \widehat{C}_{mn}^u$. Then, since $\widehat{C}_{kl}^s \subseteq \widehat{C}_{ij}^s$ are both full height vertical (stable) strips and $\widehat{C}_{kl}^u \subseteq \widehat{C}_{mn}^u$ are both full length horizontal (unstable) strips, it follows that $f^i(\widehat{C}_{kl}^s)$ is “full height” relative to the horizontal strip \widehat{C}_{ij}^u and $f^{-m}(\widehat{C}_{kl}^u)$ is “full width” relative to the vertical strip \widehat{C}_{mn}^s . Since $f^i(\widehat{C}_{kl}^s) = f^{-m}(\widehat{C}_{kl}^u)$ we complete the proof. \square

Proof of Lemma 11.14. Let $m = k - i$. Directly from the definitions,

$$(11.13) \quad C_{kl}^s = f^{-k}(\widehat{C}_{kl}^u \cap C^s) = f^{-i}(f^{-m}(\widehat{C}_{kl}^u \cap C^s)).$$

From Sublemma 11.16 we have $f^{-m}(\widehat{C}_{kl}^u) = \widehat{C}_{ij}^u \cap \widehat{C}_{mn}^s$ and thus (11.11) gives

$$f^{-m}(\widehat{C}_{kl}^u \cap C^s) = \widehat{C}_{ij}^u \cap \widehat{C}_{mn}^s \cap f^{-m}(C^s) = \widehat{C}_{ij}^u \cap C_{mn}^s.$$

Substituting this into (11.13) and using the saturation condition which implies $C_{mn}^s \subseteq C^s$, we get $C_{kl}^s = f^{-i}(\widehat{C}_{ij}^u \cap C_{mn}^s) \subseteq f^{-i}(\widehat{C}_{ij}^u \cap C^s) =: C_{ij}^s$, which implies the statement. \square

Proof of Proposition 11.4. Since the family of sets $\{\Gamma_{ij}^s\}_{ij \in I}$ are pairwise either nested or disjoint, they are partially ordered by inclusion. We can therefore define the set $I^* \subset I$ of indices ij which are *maximal* with respect to this partial order. We then let $\mathcal{P} := \{\Gamma_{ij}^s\}_{ij \in I^*}$. By Lemma 11.13, every point $x \in \Gamma^s$ belongs to some Γ_{ij}^s for some $ij \in I$ and therefore must also belong to some maximal element Γ_{ij}^s for some $ij \in I^*$. Thus \mathcal{P} is a partition of Γ into *pairwise disjoint* s -subsets whose images are u -subsets. This gives the Markov–Young structure. To see that it is a First Return Topological Young Tower we suppose by contradiction that there exists some $kl \in I^*$, $x \in \Gamma_{kl}^s$ and $0 < i < k$ such that $f^i(x) \in \Gamma$. Then Lemma 11.14 implies that there exists some $ij \in I$ such that $\Gamma_{kl}^s \subset \Gamma_{ij}^s$, contradicting the maximality of Γ_{kl}^s . \square

11.4. Fat rectangles and Young towers. In this section we prove Proposition 11.5. We split the proof into two independent parts, one to prove the hyperbolicity and distortion conditions **(Y1)**–**(Y2)** which follow from the hyperbolicity and the fatness condition, and the second to prove the integrability of the return times, which follows from the strong T-return property.

11.4.1. Hyperbolicity and distortion properties of the tower. We will verify Conditions **(Y1)** and **(Y2)** in Definition 2.8. Fix $i \in \mathbb{N}_T$, $j \in \{1, \dots, \kappa_i\}$, and $x \in \Gamma_{ij}^s$. Condition **(Y1)**(a) follows from the fact that the map $F = f^{Ti}: \Gamma_{ij}^s \rightarrow \Gamma_{ij}^u$ has the (C, λ) -hyperbolic branch property with constant $C > 0$ and $0 < \lambda < 1$ independent of x . Condition **(Y1)**(b) can be shown by a similar argument.

We now prove Condition **(Y2)**(a), the proof of Condition **(Y2)**(b) is similar. It suffices to show that for any $z \in \Gamma$ and $w \in V_z^s$ we have

$$(11.14) \quad \left| \log \frac{\text{Jac}^u F(z)}{\text{Jac}^u F(w)} \right| \leq cd(z, w)^{\alpha_1}$$

for some $c > 0$ and $\alpha_1 > 0$. Indeed, setting $z = F^n(x)$ and $w = F^n(y)$, the desired bounded distortion estimate follows from (11.14) and Condition **(Y1)**(a).

To show (11.14) notice that $x \in \Gamma_{ij}^s$ for some $i \in \mathbb{N}_T$ and $j \in \{1, \dots, \kappa_i\}$ and hence,

$$(11.15) \quad \left| \log \frac{\text{Jac}^u F(z)}{\text{Jac}^u F(w)} \right| = \left| \sum_{p=0}^{Ti-1} \log \frac{\text{Jac}^u f(f^p(z))}{\text{Jac}^u f(f^p(w))} \right|.$$

Since $z \in \Lambda_\ell$ and $w \in V_z^s$, for $0 \leq p \leq Ti - 1$ we have for some $C_1 > 0$ that $f^p(w) \in V_{f^p(z)}^s$ and

$$(11.16) \quad d(f^p(z), f^p(w)) \leq C_1 e^{\epsilon p} \lambda^p d(z, w).$$

Furthermore, by Proposition 8.3 (see (8.6)), we have for some $C_2 > 0$ and $\beta > 0$ that

$$(11.17) \quad d(E_{f^p(z)}^s, E_{f^p(w)}^s) \leq C_2 d(f^p(z), f^p(w))^\beta.$$

Since f is $C^{1+\alpha}$, it follows from (11.17) and (11.16) that for some $C_3 > 0$ and $C_4 > 0$, we have

$$\begin{aligned} \left| \frac{\text{Jac}^u f(f^p(z))}{\text{Jac}^u f(f^p(w))} - 1 \right| &= \left| \frac{\text{Jac}^u f(f^p(z)) - \text{Jac}^u f(f^p(w))}{\text{Jac}^u f(f^p(w))} \right| \\ &\leq C_3 (d((f^p(z), f^p(w))^\alpha + d((f^p(z), f^p(w))^{\alpha_2})) \\ &\leq C_4 e^{\epsilon \alpha_2 p} \lambda^{\alpha_2 p} d(z, w)^{\alpha_2} \end{aligned}$$

for some $\alpha_2 > 0$. The estimate (11.14) now follows from (11.15).

11.4.2. Integrability of the return times. Let $F: \Gamma \rightarrow \Gamma$ be the induced map to the base of the Topological First T-Return Young Tower where $F(x) = f^{\tau(x)}(x)$ and $\tau(x)$ is the first return time to Γ which is a multiple of T . To simplify the notation *we will assume here that $T = 1$* since the integrability property is not affected by taking fixed multiples of the return time, recall Definition 2.9. For every $n \geq 0$ let

$$(11.18) \quad R_n(x) := \sum_{j=0}^{n-1} \tau(F^j(x)) \quad \text{and} \quad v_n(x) := \#\{0 < i \leq n : f^i(x) \in \Gamma\}$$

where we define $R_0(x) = 0$ by convention. Notice that R_n, v_n are on quite different time scales, the index n in v_n refers to the iterates of the original map f , whereas in R_n it refers to the iterates of the induced map F . The following relation between the two quantities is not surprising but neither is it completely trivial, we thank Vilton Pinheiro for explaining it to us.

Lemma 11.17 (Pinheiro [53]). *Let $x \in \Gamma$ and suppose $\lim_{n \rightarrow \infty} R_n(x)/n$ exists.*

Then

$$\lim_{n \rightarrow \infty} \frac{v_n(x)}{n} = \left(\lim_{n \rightarrow \infty} \frac{R_n(x)}{n} \right)^{-1}.$$

In particular $\lim_{n \rightarrow \infty} v_n(x)/n$ exists and is equal to 0 if $\lim_{n \rightarrow \infty} R_n(x)/n = \infty$.

Proof of Lemma 11.17. For any $x \in \Gamma$, by definition of $v_n(x)$ we have

$$v_n(x) := \#\{0 < i \leq n : f^i(x) \in \Gamma\} = \max\{k \geq 0 : \sum_{j=0}^{k-1} \tau(F^j(x)) \leq n\}$$

and so, for every $n \geq \tau(x)$, so that $v_n(x) \geq 1$, we have

$$R_{v_n(x)}(x) := \sum_{j=0}^{v_n(x)-1} \tau(F^j(x)) \leq n < \sum_{j=0}^{v_n(x)} \tau(F^j(x)) =: R_{v_n(x)+1}(x).$$

Dividing through by $v_n(x)$ this gives

$$(11.19) \quad \frac{R_{v_n(x)}(x)}{v_n(x)} \leq \frac{n}{v_n(x)} < \frac{v_n(x) + 1}{v_n(x)} \frac{R_{v_n(x)+1}(x)}{v_n(x) + 1}$$

Since $(v_n(x) + 1)/v_n(x) \rightarrow 1$ as $n \rightarrow \infty$ and the limit of the sequence $R_n(x)/n$ exists, the subsequences on the left and right hand side of (11.19) also converge to the same limit. It follows that the $n/v_n(x)$ converges and therefore also $v_n(x)/n$ to a limit as in the statement. \square

Lemma 11.17 easily implies the integrability of the return times. Indeed, by the results in [62] the induced map $F : \Gamma \rightarrow \Gamma$ admits an SRB measure $\hat{\mu}$ whose conditional measures $\hat{\mu}_z$ on unstable curves of points of Γ are equivalent to the Lebesgue measure $m_{V_z^u}$ on these same curves. It is therefore sufficient to show the integrability with respect to one of these conditional measures. By the invariance of $\hat{\mu}$ it follows that for $\hat{\mu}_z$ almost every x the Birkhoff averages of the return time τ converge to the integral of τ , i.e.

$$\int \tau d\hat{\mu}_z = \lim_{n \rightarrow \infty} \frac{R_n(x)}{n}$$

where both quantities are in principle allowed to be infinite. It is therefore sufficient to prove that the limit on the right hand side is finite. Supposing that it is infinite, by Lemma 11.17 this would imply that $v_n(x) = 0$ but this contradicts the strong T -return property. This completes the proof of the integrability of the return times and thus the proof of Theorem B.

12. HYPERBOLIC MEASURES HAVE NICE REGULAR SETS: PROOF OF THEOREM C

In this section we prove Theorem C. The non-trivial part of the proof is to show that we can find arbitrarily small domains Γ_{pq} with $\mu(\Gamma_{pq} \cap \Lambda_\ell) > 0$ where $p, q \in \Lambda_\ell$ are periodic points. Then letting $T > 0$ be any common multiple of the periods of p and q , it follows that p, q are fixed points for f^T and therefore Γ_{pq} is a nice domain with $T(\Gamma_{pq}) = T$. Moreover, μ is also f^T -invariant and therefore μ -a.e. $x \in A := \Gamma_{pq} \cap \Lambda_\ell$ returns to A with positive frequency for iterates which are multiples of T , in both forward and backward time. Thus A satisfies the strong T -return property, and in particular the T -return property. If μ is an SRB measure it follows by definition that A is fat. Thus Theorem C follows from the statement below.

Proposition 12.1. *Let f be a $C^{1+\alpha}$ diffeomorphism, μ an ergodic non-atomic χ -hyperbolic measure, and Λ a χ -hyperbolic set. Fix $\lambda \in (0, \chi)$ and $\epsilon \in (0, \epsilon_1(f, \chi, \lambda))$. Let $U \subset M$ be an open set and $\ell \in \mathbb{N}$ such that $\mu(U \cap \Lambda_\ell) > 0$. Then with ℓ' as in Theorem F, there are $(\lambda/4, 2\epsilon, \ell + \ell')$ -regular periodic points p, q such that Γ_{pq} is defined, contained in U , and satisfies $\mu(\Gamma_{pq} \cap \Lambda_\ell) > 0$. Since $\Gamma_{pq} \subset U$, $\text{diam } \Gamma_{pq}$ can be made arbitrarily small.*

Before proving Proposition 12.1, we use Theorem F to establish a result reminiscent of the Katok Closing Lemma. Say that $y \in B(x, \delta) \cap \Lambda_\ell$ is Λ_ℓ -nonwandering if there is a sequence $n_k \rightarrow \infty$ and $y_k \in \Lambda_\ell \cap f^{-n_k} \Lambda_\ell$ such

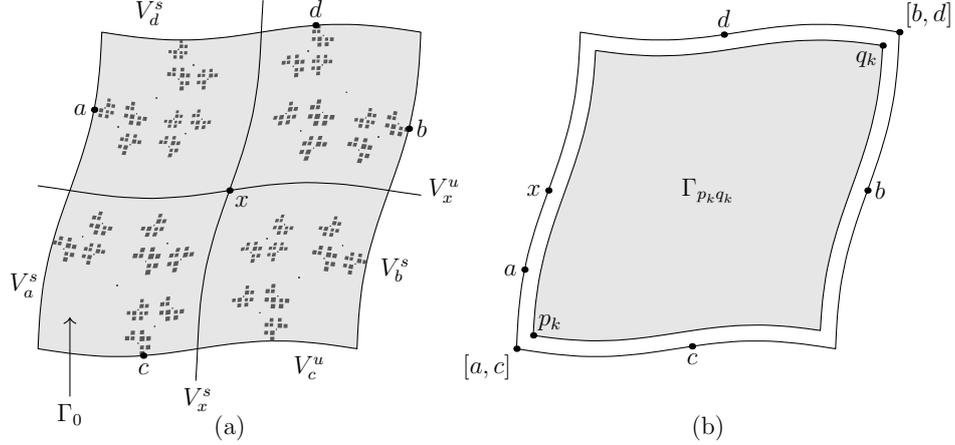


FIGURE 12.1. Proving Proposition 12.1.

that $y_k, f^{n_k}(y_k) \xrightarrow{k \rightarrow \infty} y$. Observe that by Poincaré Recurrence, every point in $\text{supp}(\mu|_{\Lambda_\ell})$ is Λ_ℓ -nonwandering.

Lemma 12.2. *Given $\delta > 0$ as in Theorem E, $\ell' \in \mathbb{N}$ as in Theorem F, and any $\ell \in \mathbb{N}$, for all Λ_ℓ -nonwandering points $y, z \in B(x, \delta e^{-\lambda \ell}/3) \cap \Lambda_\ell$ there is a sequence of $(\lambda/4, 2\epsilon, \ell + \ell')$ -regular periodic points $p_k \xrightarrow{k \rightarrow \infty} V_y^s \cap V_z^u$.*

Proof. Suppose y, z are as in the hypothesis. Choose $n_k \rightarrow \infty$ and $y_k \in \Lambda_\ell \cap f^{-n_k} \Lambda_\ell$ such that $y_k, f^{n_k}(y_k) \rightarrow y$. Choose m_k, z_k similarly for z . For suitably large k , we have

$$y_k, f^{n_k}(y_k), z_k, f^{m_k}(z_k) \in B(x, \delta e^{-\lambda \ell}/2)$$

and thus in particular

$$d(f^{n_k}(y_k), z_k) \leq \delta e^{-\lambda \ell} \quad \text{and} \quad d(f^{m_k}(z_k), y_k) \leq \delta e^{-\lambda \ell}.$$

It follows that $y_k, f(y_k), \dots, f^{n_k-1}(y_k), z_k, f(z_k), \dots, f^{m_k-1}(z_k), y_k$ is a $(\bar{\ell}, \delta, \lambda)$ -pseudo-orbit with

$$\ell_i = \ell + \begin{cases} \min(i, n_k - i) & 0 \leq i \leq n_k, \\ \min(i - n_k, n_k + m_k - i) & n_k \leq i \leq n_k + m_k. \end{cases}$$

Repeating this finite pseudo-orbit \bar{x} periodically gives a periodic bi-infinite pseudo-orbit to which we can apply Theorem F and obtain a $(\lambda/4, 2\epsilon, \ell + \ell')$ -regular periodic shadowing point p_k . Note that $p_k \in \mathcal{N}_{\bar{x}}^0 \cap \mathcal{N}_{\bar{x}}^{n_k+m_k}$, and that the intersections converge to $V_y^s \cap V_z^u$ as $k \rightarrow \infty$ because $y_k \in \mathcal{N}_{\bar{x}}^0$ and $f^{m_k} z_k \in \mathcal{N}_{\bar{x}}^{n_k+m_k}$. Thus $p_k \rightarrow V_y^s \cap V_z^u$. \square

Proof of Proposition 12.1. Fix $x \in U \cap \text{supp}(\mu|_{\Lambda_\ell})$. Since Λ_ℓ is closed, we have $\text{supp}(\mu|_{\Lambda_\ell}) \subset \Lambda_\ell$. Choose $\delta', \delta > 0$ sufficiently small that $B(x, \delta') \subset U$, and such that for every $y, z \in \overline{B(x, \delta)} \cap \Lambda_\ell$, the intersection $V_y^s \cap V_z^u$ is a single point and lies in $B(x, \delta') \cap \Lambda_{\ell'}$. Assume also that δ is chosen small enough and ℓ' large enough to satisfy Lemma 12.2.

Let $Z := \overline{B(x, \delta)} \cap \text{supp}(\mu|_{\Lambda_\ell})$. Observe that Z is compact, and that $\mu(Z) > 0$ by our choice of x . Let $\pi^s: Z \rightarrow V_x^u$ and $\pi^u: Z \rightarrow V_x^s$ be projection along local stable and unstable leaves, respectively. Since $V_x^{s,u}$ are one-dimensional we can equip each with a total order, and by compactness we can choose $a, b, c, d \in Z$ such that

$$\begin{aligned}\pi^s(a) &= \inf \pi^s(Z), & \pi^s(b) &= \sup \pi^s(Z), \\ \pi^u(c) &= \inf \pi^u(Z), & \pi^u(d) &= \sup \pi^u(Z).\end{aligned}$$

Let Γ_0 be the region bounded by V_a^s , V_b^s , V_c^u , and V_d^u , as shown in Figure 12.1. Observe that $\Gamma_0 \supset Z$ and thus $\mu(\Gamma_0 \cap \Lambda_\ell) > 0$. By Lemma 12.2 there are periodic points $p_k, q_k \in \Lambda_{\ell'}$ such that $p_k \rightarrow V_a^s \cap V_c^u$ and $q_k \rightarrow V_b^s \cap V_d^u$. It is possible that none of the domains $\Gamma_{p_k q_k}$ contains x (this can occur, for example, if $a \in V_x^u$, as in Figure 12.1(b)); on the other hand, the union $\bigcup_n \Gamma_{p_n q_n}$ covers all of Z except possibly for $Z \cap V_a^s \cup V_b^s \cup V_c^u \cup V_d^u$. Since μ is non-atomic, a single local stable or unstable curve always has zero measure, thus this subset is μ -null. Using the fact that $\mu(Z) > 0$, we conclude that there is some n such that $\mu(\Gamma_{p_n q_n} \cap Z) > 0$. This completes the proof of the proposition. \square

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