SRB measures, unstable manifolds, and effective hyperbolicity

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M a cpt mfd, $U \subset M$ open, $f: \bar{U} \to U$ local diffeo ($C^{1+\alpha}$)

* If ~~trajectories~~ separate exponentially quickly, then initial error blows up, predictions worthless. Treat (M, f) as stochastic process: $\varphi: U \to \mathbb{R}$ an observable, consider $\varphi, \varphi \circ f, \varphi \circ f^2, \ldots$ NEED AN INVARIANT MEASURE.

Ergodic theorem gives LLN for μ-a.e. ~~trajectory~~ if μ ergodic.

1. Is "μ-a.e." big?
2. What other statistical properties? CLT? EDC? LDP?

Focus on ①.

- natural notion of a.e. is Lebesgue-a.e.
- get this if $\mu \ll Leb$.
- often f is dissipative, $\Lambda = \bigcap_{n \geq 1} f^n(U)$ has volume 0.
- motivates defn of SRB measure
  - hyperbolic (nonzero Lyap exp)
  - conditionals μ_W on unstables satisfy $\mu_W \ll m_W$ (leaf volume)

[Q] When does f have an SRB measure?

Natural condition: pos. Leb. mass set of pts with nonzero exponents
- Necessary but unclear if sufficient

Natural approach: $\mu_n = \frac{1}{n} \sum_{k=0}^{n-1} f_*^k Leb$, $\mu_{n_k} \to \mu \in \mathcal{M}(\Lambda, f)$
- problem: neither $\{\mu \mid \lambda(\mu) \neq 0\}$ nor $\{\mu \mid \mu_{W^u} \ll m_{W^u}\}$ is compact in $\mathcal{M}(U)$.

① weaker dynamics: $|Df_x^n(v)| \geq C e^{\lambda n} |v|$, $C = C(x)$, $\lambda = \lambda(x)$

may have $C, \lambda \to 0$

② weaker geometry: W^u may degenerate in size/curvature
- also, density fn may degenerate (L^1 not cpt)

If expansion is controlled, densities will follow - so challenge is dynamics & geometry.

$\boxed{\text{Rmk}}$ Known: Anosov / Axiom A (Sinai, Ruelle, Bowen)

- uses Markov partitions, uniformity of dynamics & geometry

PH ~~with~~ NU exp/contr E^c (Alves, Bonatti, Viana)

- uses uniformity of geometry

small perturbations of one-dim maps (Jakobson, Benedicks, Carleson, Young, Wang)

- uses Young towers / inducing schemes, stronger conditions than seem necessary for just SRB (also get stat prop)

Let's make above approach more precise. Spse $\exists S \subset U$, Leb $S > 0$, $T_x M = E^s \oplus E^u$ $\forall x \in S$, $\dim E^u = d$

$\overline{\lim} \frac{1}{n} \log \| Df_x^n(v^s)\| < 0$,

$\underline{\lim} \frac{1}{n} \log \| Df_x^n(v^u)\| > 0$. $\Leftarrow (x \in H)$

Let $\mathcal{R} = \{W \subset U \mid W \ni x \in S,\ W \approx E^u(x)\}$, so $f: \mathcal{R} \circlearrowleft$

& let $\mathcal{R}' = \{(W,\rho) \mid W \in \mathcal{R},\ \rho \in L^1(m_W)\}$

Now $\Phi: \mathcal{R}' \to \mathcal{M}(U)$

$(W,\rho) \mapsto \int \bullet\, \rho(x)\, dm_W(x)$

& $\Phi^*: \mathcal{M}(\mathcal{R}') \to \mathcal{M}(U)$ give notion of "a.c. on W^u" - meas.

$\boxed{\mathcal{M}^{ac} \cap \mathcal{M}^h \cap \mathcal{M}(\Lambda, f) = \{SRB\}}$

and we see that $\mu_n \in \mathcal{M}^{ac}$ $\forall n$. Also $\mu_n \in \mathcal{M}^h = \{\mu \mid \mu(H) = 1\}$. However, neither $\mathcal{M}^{ac}$ nor $\mathcal{M}^h$ is wk*-cpt, so why should $\mu = \lim \mu_{n_k}$ be in them?

Usual story in NUH $\to$ trade invariance for cptness

- Pesin sets: Y inv, not cpt, Y_ℓ cpt, not inv, but NUH meas μ has $\mu(Y_\ell) \to 1$.
- we want to play a similar game with $\mathcal{M}^{ac}$, $\mathcal{M}^h$

$\boxed{\text{GOAL}}$ ① cpt sets ~~$\mathcal{M}^{ac}_K$, $\mathcal{M}^h_K$~~ $\mathcal{M}^{ac,h}_K \subset \mathcal{M}^{ac} \cap \mathcal{M}^h$

② criterion on $x \in S$ guaranteeing that $\mu_n(\mathcal{M}^{ac,h}_K) \gg 0$

- requires $f^n(W^u)$ to contain large pieces of good mfds w/ unif controlled dynamics.

$K = (\theta, \gamma, \varkappa, r, C, \lambda, \beta, L)$

geometry: $P_K = \{ \exp_x \text{graph}\, \psi \mid x \in \mathcal{U},\ T_x M = G \oplus F,\ \measuredangle(G,F) \geq \theta,$
$\psi \in C^{1+\alpha}(B_G(r), F),\ \psi(0)=0,\ D\psi(0)=0,$
$\|D\psi\| \leq \gamma,\ |D\psi|_\alpha \leq \varkappa \}$

dynamics: (in E^u) $Q_{K,N} = \{ W \subset \mathcal{U} \mid \forall\, y,z \in W$, we have
$d(f^{-j}y, f^{-j}z) \leq C e^{-\lambda j} d(y,z)\ \forall\, 0 \leq j \leq N \}$

(in E^s) $H_{K,N}(W) = \{ y \in W \mid T_y M = (T_y W) \oplus G,$
$\|Df_y^{-j}(v)\| \geq C^{-1} e^{\lambda j} \|v\|\ \forall v \in G,\ 0 \leq j \leq N \}$

$R_{K,N} = \{ W \in P_K \cap Q_{K,N} \mid m_W(H_{K,N}(W)) \geq \beta\, m_W(W) \}$ ← admissibles

$R_K = \cap_N R_{K,N}$ ← unstables
$R = \cup_K R_K$

$\boxed{\mathcal{M}^{ac,h}_{K,N} = \Phi^*(\mathcal{M}(R_{K,N}))} \rightsquigarrow \mathcal{M}^{ac,h}_K = \cap_N \mathcal{M}^{ac,h}_{K,N} \subset \mathcal{M}^{ac} \cap \mathcal{M}^h$

Thm (C-Dolgopyat-Pesin)
Let $M, \mathcal{U}, f$ as before, $\mu_n \in \mathcal{M}(\mathcal{U})$, $\mu_n \to \mu \in \mathcal{M}(f)$.
Sps $\exists\, K$, $q_n \to \infty$, $\nu_n \in \mathcal{M}^{ac,h}_{K,q_n}$ s.t.
$\mu_n \geq \nu_n$ & $\|\nu_n\| \geq \delta > 0$.
Then some ergodic component of μ is an SRB measure.

How to produce ν_n? Need pairs $(x,n) \in \mathcal{U} \times \mathbb{N}$ s.t.
$f^n(W^u) \in R_{K,N}$. Controlling dynamics suggests notion of hyperbolic time - controlling geometry requires introduction of effective hyperbolicity.

$\sqrt{}$ forward invariant

Sps $\Lambda \subset U$ has invariant cone families $K^{s,u}$. (maximal)

$\lambda^u(x) = \inf \log \| Df(v^u) \|$

$\lambda^s(x) = \sup \log \| Df(v^s) \|$

$\Delta(x) = \frac{1}{2} \max(0, \lambda^s(x) - \lambda^u(x))$ ← defect from domination

Need

(#) $\boxed{\varliminf \frac{1}{n} \sum_{k=0}^{n-1} [\lambda^u(f^k x) - \Delta(f^k x)] > 0}$

Consider $\Gamma_\lambda^e(x) = \{ n \in \mathbb{N} \mid \sum_{j=k}^{n-1} [\lambda^u - \Delta] \geq \lambda(n-k) \ \forall\ 0 \leq k < n \}$

(effective hyperbolic times)

Pliss' lemma: (#) $\Rightarrow \underline{\delta}(\Gamma_\lambda^e(x)) > 0$. (lower asymptotic density)

Let $\theta(x) = \measuredangle(K^s(x), K^u(x))$, & suppose

($\flat$) $\boxed{\lim_{\bar\theta \to 0} \bar\delta \{ n \mid \theta(f^n x) < \bar\theta \} = 0}$

[Thm] (C.-Pesin)

If (#) & (b) hold, then $\forall\ \varepsilon > 0\ \exists\ \Gamma_\varepsilon \subset \mathbb{N}$ s.t. $\bar\delta(\Gamma_\varepsilon) < \varepsilon$

& $\forall\ n \in \Gamma_\lambda^\varepsilon(x) \setminus \Gamma_\varepsilon(x)$, we have $V \ni x$ s.t.

$$\boxed{f^n(V) \in P_K \cap Q_{K,n}}$$

(In fact, any admissible mfd ~~through~~ x contains such a V).

[Rmk] This allows us to prove a non-uniform finite-information closing / shadowing lemma.

Only remains to get condition on $E^s(x)$.

$\Gamma^s_{C,\lambda,q}(x) = \{ n \in \mathbb{N} \mid \forall\, v \in E^s(f^k x),\ \ k \in [n-q, n],$
we have $\|Df^{n-k}(v)\| \le C e^{-\lambda(n-k)} \|v\| \}$

[NB] Similar to notion of hyp time, but weaker

Let $S = \{ x \in A \mid \Gamma^e_\lambda(x) \cap \Gamma^s_{C,\lambda,q}(x)$ has $\underline{\delta} > 0$
& (b) holds $\}$

[Thm] (C.-Dolgopyat-Pesin)
If Leb$(S) > 0$, ~~or $m_W(S_W) > 0$ for some~~
then f has an SRB measure.
Similarly, if $m_W(S_W) > 0$
($S_W = \{ x \in S \cap W \mid T_x W \subset K^u(x) \}$, $\dim W = \dim K^u$)
then f has an SRB measure.

[Rmk] Gives alternate pf in uniform case. In some examples can verify that each of Γ^e_λ & $\Gamma^s_{C,\lambda,q}$ have $\underline{\delta} = 1$.

[Eg] Dissipative Axiom A, do Katok slowdown near a fixed pt. Can also add shear so that geometry is non-uniform.