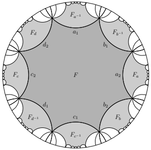
Counting closed geodesics on surfaces without conjugate points

Vaughn Climenhaga

University of Houston

April 30, 2021

Joint work with Gerhard Knieper (Bochum) and Khadim War (IMPA)



M = closed connected surface with genus ≥ 2

$G(t) = \{$closed geodesics on M with length $\leq t\}$

What can we say about $\#G(t)$?

Negative curvature (Margulis 1970): $\#G(t) \sim \frac{e^{ht}}{ht}$ for some $h > 0$

- $f(t) \sim g(t)$ means $\frac{f(t)}{g(t)} \to 1$ as $t \to \infty$ (any dim)

In general, can have continuum of closed geodesics (flat cylinder), so let $P(t) = \{$free homotopy classes in $G(t)\}$

p, q in universal cover X are **conjugate** if joined by > 1 geodesic

- curvature $\leq 0 \Rightarrow$ no conjugate points ($\not\Leftarrow$)

Theorem (C., Knieper, War, arXiv:2008.02249)

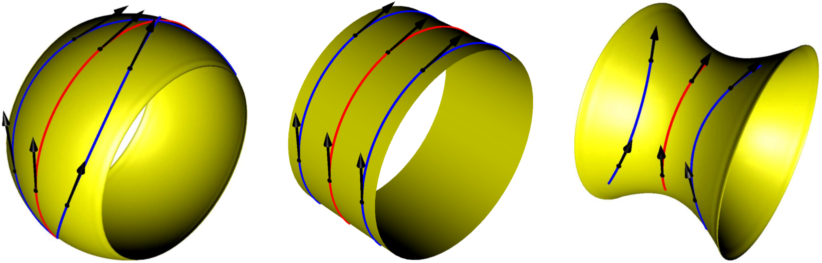
No conjugate points $\Rightarrow \#P(t) \sim \frac{e^{ht}}{ht}$ *(dim 2, some higher-dim)*

From geometry to dynamics; geodesic flow and curvature

$\phi^t : SM \to SM$ geodesic flow on unit tangent bundle

$$v \in SM \;\rightsquigarrow\; c_v \text{ geodesic with } \dot{c}_v(0) = v \;\rightsquigarrow\; \phi^t(v) := \dot{c}_v(t)$$

Closed geodesics $\leftrightarrow$ periodic orbits for geodesic flow

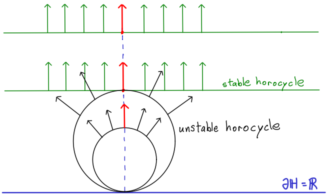


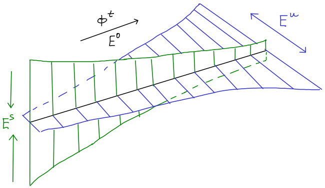
$K > 0$ $\qquad$ $K = 0$ $\qquad$ $K < 0$

Constant negative curvature and scaling of leaf measures

Constant negative curvature: universal cover X is hyperbolic plane $\mathbb{H} = \{x + iy : y > 0\}$ with Riemannian metric proportional to $\frac{\mathrm{Euc}}{y}$

Normal vector fields to horocycles are uniformly contracted by $\phi^{\pm t}$, giving an Anosov splitting $TS\mathbb{H} = E^u \oplus E^s \oplus E^0$ (flow direction)





Let m^s, m^u be Lebesgue measure along stable/unstable leaves, then

$$m^u(\phi^t A) = e^{ht} m^u(A) \quad \text{and} \quad m^s(\phi^t A) = e^{-ht} m^s(A)$$

The product $m^u \times m^s \times \mathrm{Leb}$ gives Liouville measure on SM.

Margulis leaf measures

Variable negative curvature still gives a topologically mixing Anosov flow, but Lebesgue measure may not scale by

$$m^u(\phi^t A) = e^{ht} m^u(A) \quad \text{and} \quad m^s(\phi^t A) = e^{-ht} m^s(A) \qquad (\star)$$

For **any** Anosov flow, Margulis built m^u, m^s satisfying $(\star)$, where now h is topological entropy (growth rate of (t, ϵ)-separated set)

- Fixed point argument on an appropriate space (Margulis 1970)
- Can also use Hausdorff measure in appropriate metric (Hamenstädt 1989, Hasselblatt 1989, ETDS)
- Interpretation via Bowen's alternate definition of entropy (C.–Pesin–Zelerowicz BAMS 2019, also C. arXiv:2009.09260)

$m = m^u \times m^s \times$ Leb is flow-invariant Bowen–Margulis measure

Properties of Bowen–Margulis measure

$$m^u(\phi^t A) = e^{ht} m^u(A) \quad \text{and} \quad m^s(\phi^t A) = e^{-ht} m^s(A) \qquad (\star)$$

For a topologically mixing Anosov flow, the Bowen–Margulis measure $m = m^u \times m^s \times \text{Leb}$ has the following properties.

- Mixing (can use Hopf argument and product structure)
- Unique measure of maximal entropy (Adler, Weiss, Bowen)
- Equidistribution: given $\epsilon > 0$, let

$$C(t) = \{\text{periodic orbits with period in } (t-\epsilon, t]\}$$

$$\nu_t = \frac{1}{\#C(t)} \sum_{c \in C(t)} \frac{1}{t} \text{Leb}_c$$

Periodic orbit measures $\nu_t \xrightarrow{\text{weak*}} m$ as $t \to \infty$

(Equidistribution follows from uniqueness if periodic orbits are separated and $\lim_{t\to\infty} \frac{1}{t} \log \#C(t) = h$)

Three levels of counting estimates

$P(t) = \{\text{periodic orbits with period} \leq t\}$
$N(t) = \text{cardinality of a maximal } (t,\epsilon)\text{-separated set}$

Growth rate: the closing lemma gives

$$h = \lim_{t\to\infty} \frac{1}{t} \log N(t) = \lim_{t\to\infty} \frac{1}{t} \log \#P(t) \qquad (\#P(t) = (\mathrm{subexp}) \cdot e^{ht})$$

Uniform counting estimates: $\left(a_{k+n} \leq a_k + a_n \Rightarrow \frac{a_n}{n} \to \inf \frac{a_n}{n}\right)$
$N(s+t) = C^{\pm 1} N(s)N(t)$ ("quasi-sub/supermultiplicative") gives

$$Ae^{ht} \leq N(t) \leq Be^{ht} \quad \Rightarrow \quad \frac{A'}{t} e^{ht} \leq \#P(t) \leq \frac{B'}{t} e^{ht}$$

Crucial for proof that m is the **unique** MME.

Margulis estimates: $\#P(t) \sim \frac{e^{ht}}{ht}$, ie., $A', B' \to \frac{1}{h}$ as $t \to \infty$

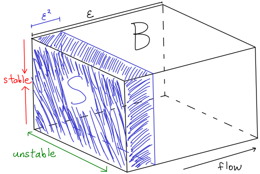
Sketch of (modified) proof of Margulis estimates

Eventual goal: Estimate the cardinality of

$$C(t) = \{\textit{periodic orbits with period in } (t-\epsilon, t]\}$$

and sum to get cardinality of $P(t)$ (becomes integral as $\epsilon \to 0$). Use periodic orbit measures ν_t and Bowen–Margulis measure m.

Step 1. Use local product structure to define flow box B with depth ϵ (in flow direction) and slice/slab S with depth ϵ^2

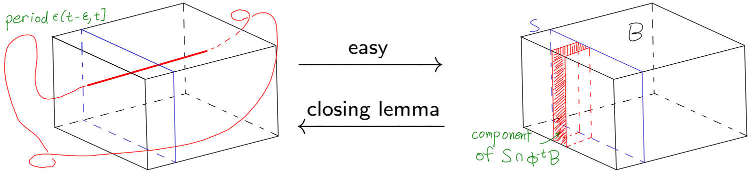


We will study the quantity

$$\nu_t(B) = \frac{1}{t\#C(t)} \sum_{c \in C(t)} \mathrm{Leb}_c(B)$$
$$= \frac{\epsilon}{t\#C(t)} \sum_{c \in C(t)} \textit{(number of times } c \textit{ crosses } B\textit{)}$$

Closing lemma and components of intersection (ν_t)

Goal: Estimate $\#C(t) = \#\{$per. orbits with period in $(t-\epsilon, t]\}$
via $\nu_t(B) = \frac{\epsilon}{t\#C(t)} \sum_{c \in C(t)}$ (number of times c crosses B)



Step 2. Let $\Gamma(t) = \{$connected components of $S \cap \phi^{-t}B\}$. The closing lemma gives a correspondence between $\Gamma(t)$ and the orbit segments in which an element of c crosses B. Thus

$$\nu_t(B) \approx \frac{\epsilon}{t} \frac{\#\Gamma(t)}{\#C(t)}$$

Scaling of leaf measures, and mixing property (m)

$C(t) = \{\text{per. orbits with per. in } (t-\epsilon, t]\}$

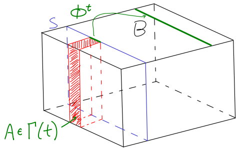
$\Gamma(t) = \{\text{conn. comp. of } S \cap \phi^{-t} B\}$

$$\nu_t(B) \approx \frac{\epsilon}{t} \frac{\#\Gamma(t)}{\#C(t)}$$

Now we estimate $m(S \cap \phi^{-t} B)$ in two different ways...

Step 3. $m = m^u \times m^s \times \text{Leb}$ and $m^{u,s}$ scale by e^{ht}, so nearly every component A in $\Gamma(t)$ has $m(A) = e^{-ht} m(S)$, giving

$$m(S \cap \phi^{-t} B) \approx \#\Gamma(t) e^{-ht} m(S)$$



Step 4. m is mixing, so $m(S \cap \phi^{-t} B) \to m(S) m(B)$, giving

$$m(S)m(B) \approx \#\Gamma(t) e^{-ht} m(S) \quad \Rightarrow \quad m(B) \approx e^{-ht} \#\Gamma(t)$$

Equidistribution (both ν_t and m)

$C(t) = \{\text{per. orbits with per. in } (t-\epsilon, t]\}$

$\Gamma(t) = \{\text{conn. comp. of } S \cap \phi^{-t} B\}$

$$\nu_t(B) \approx \frac{\epsilon}{t} \frac{\#\Gamma(t)}{\#C(t)}$$

$$m(B) \approx e^{-ht} \#\Gamma(t) \quad \Rightarrow \quad \#\Gamma(t) \approx m(B) e^{ht}$$

Preliminary estimate of $\#C(t)$ by combining the above:

$$\#C(t) \approx \frac{\epsilon}{t} \frac{\#\Gamma(t)}{\nu_t(B)} \approx \frac{\epsilon}{t} \frac{m(B)}{\nu_t(B)} e^{ht} \Rightarrow \lim_{t\to\infty} \frac{1}{t} \log \#C(t) = h \quad (1)$$

Step 5. General argument as in proof of variational principle uses this estimate to show that every limit point of $(\nu_t)_{t\to\infty}$ is an MME, and uniqueness gives equidistribution result $\nu_t \to m$. Then the first part of (1) gives $\#C(t) \approx \frac{\epsilon}{t} e^{ht}$.

Conclusion of the proof and review of tools

$C(t) = \{\text{per. orbits with per. in } (t-\epsilon, t]\}$
$P(T) = \{\text{per. orbits with per. } \leq T\}$

$\#C(t) \approx \frac{\epsilon}{t} e^{ht}$

Step 6. Divide $(1, T]$ into ϵ-intervals $(t_k - \epsilon, t_k]$:

$$\#P(T) \approx \sum_k \#C(t_k) \approx \sum_k \epsilon \frac{e^{ht_k}}{t_k} \xrightarrow{\epsilon \to 0} \int_1^T \frac{1}{t} e^{ht}\, dt \approx \frac{e^{hT}}{hT}$$

What did we use?

- *The flow has a local product structure*
- *There are leaf measures m^s, m^u that scale by $e^{\pm ht}$*
- $m = m^s \times m^u \times \mathrm{Leb}$ *is mixing and is the unique MME*
- *Periodic orbits are ϵ-separated*

Beyond negative curvature

For geodesic flows, negative curvature implies Anosov

More general metrics require new ideas

- Flat cylinders can give continuum of closed geodesics; all in same homotopy class, so count homotopy classes instead
- Homotopy growth forces entropy (Knieper 1983), but not vice versa, eg., metrics on sphere with positive entropy
- Need restrictions on manifold if we expect counting estimates

No conjugate points – equivalent characterizations

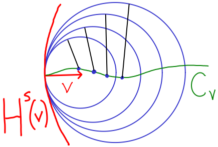
- Every $x \neq y$ in universal cover X joined by unique geodesic
- Exponential map $\exp_x : T_x X \to X$ a diffeomorphism

$\{$neg. curvature$\} \subsetneqq \{$nonpos. curvature$\} \subsetneqq \{$no conjugate points$\}$

Foliations via horospheres

M a manifold without conjugate points, X universal cover

Given $v \in SX$, can define stable horosphere



$$H^s(v) = \lim_{r\to\infty} \partial B_X(c_v(r), r)$$

where c_v is the geodesic with $\dot{c}_v(0) = v$. Normal vector field to $H^s(v)$ gives stable foliation W^s. Reverse time for unstable W^u.

Leaves may not contract, $W^{s,u}$ may not be transverse (e.g. $\mathbb{R}^2$)

Nonpositive curvature: $W^{s,u}$ are continuous, get contraction and transversality on an open and dense set if M is "rank 1"

No conjugate points: $W^{s,u}$ can be discts (Ballmann, Brin, Burns "dinosaur"), no proof of contraction/transversality on any open set

How to define the flow box B? Requires product structure...

Product structure from the boundary at infinity

Assume no conjugate points, surface with genus ≥ 2

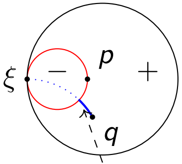
v, w on same leaf of $W^s \quad \Rightarrow \quad \sup_{t>0} d(c_v(t), c_w(t)) < \infty$

Write $c_v \sim c_w$ in this case; boundary at infinity ∂X is set of equivalence classes ("set of possible futures/pasts")

- Join all pasts/future: for all $(\xi, \eta) \in \partial^2 X = (\partial X)^2 \setminus \mathrm{diag}$, there is a geodesic c in X with $c(-\infty) = \xi$ and $c(\infty) = \eta$

Use Busemann functions, define Hopf map

$$H\colon SX \to \partial^2 X \times \mathbb{R}$$
$$v \mapsto (c_v(\pm\infty), b_{c_v(-\infty)}(\pi v, p))$$



$b_\xi(q, p) = \pm\text{length}$

A flow-invariant μ on SM gives measure $\bar{\mu}$ on $\partial^2 X$ that is invariant under action of $\Gamma = \pi_1(M)$, **and vice versa** *(pull back by H)*

Constructing conformal measures: a rough idea

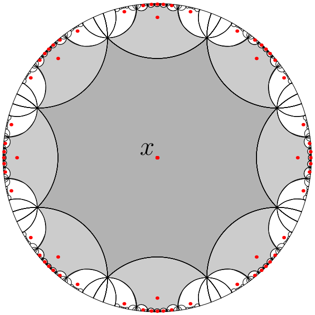
MME/Gibbs: "every orbit segment of length t gets weight e^{-ht}."

Of course this is nonsense: uncountable! Options to resolve:

1. Use periodic orbits or (t, ϵ)-separated sets (Bowen)
2. Use isometric action of $\Gamma = \pi_1(M)$ on X (Patterson–Sullivan)
   Geodesic segment corresponds to pair of points in X

Now with a countable set, can sum:

1. $\sum_{c \in \{\text{periodic orbits}\}} e^{-h \cdot \text{length}(c)} \text{Leb}_c$
2. $\sum_{\gamma \in \Gamma} e^{-hd(x,\gamma x)} \delta_{\gamma x}$ for $x \in X$



But these are infinite! Two options:

1. Finite part of sum, normalize, limit
2. Replace h with $s > h$, normalize, take $s \searrow h$

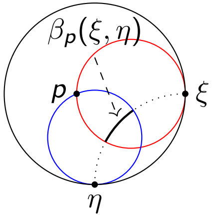
Patterson–Sullivan construction and Margulis measure

Fix reference point $x \in X$. For each $p \in X$ get conformal density

$$\nu_p = \lim_{s \searrow h} \left[\text{normalize}\left(\sum_{\gamma \in \Gamma} e^{-sd(p,\gamma x)} \delta_{\gamma x} \right) \right] \qquad (\operatorname{supp} \nu_p = \partial X)$$

Can construct a Γ-invariant probability measure $\bar{\mu}$ on $\partial^2 X$ (a geodesic current) by

$$d\bar{\mu}(\xi, \eta) = e^{h\beta_p(\xi,\eta)}\, d\nu_p(\xi)\, d\nu_p(\eta)$$



Use Hopf map to pull back to a flow-invariant measure on SM, which is the unique MME.

Scaling properties of ν_p w.r.t. p lead to Margulis relations

Prior results using Patterson–Sullivan approach

Negative curvature. Kaimanovich (1990) showed that the construction due to Patterson and Sullivan (1970s) can be used to obtain Bowen–Margulis measure.

Roblin (2003) used this approach to get Margulis estimates for closed geodesics (applies to some noncompact manifolds)

Nonpositive curvature, rank 1. Knieper (1997–98) got unique MME via Patterson–Sullivan. His proof gives uniform counting estimates for closed geodesics (level 2 of the 3-level hierarchy)

Babillot (2002) showed that the unique MME is mixing

Ricks (2019) proved Margulis counting estimates (in CAT(0))

- Defines flow box using Hopf map: $B = H^{-1}(\mathbf{P} \times \mathbf{F} \times [0, \epsilon])$ where $\mathbf{P}, \mathbf{F}$ are disjoint neighborhoods in ∂X

New challenges for manifolds with no conjugate points

No conjugate points. *Desired ingredients:*

- *Periodic orbits are ϵ-separated* *(Count free homotopy classes)*
- *Product structure for flow* *(Provided by ∂X and Hopf map)*
- *Leaf measures m^s, m^u that scale by $e^{\pm ht}$* *(Patterson–Sullivan)*
- $m = m^s \times m^u \times \mathrm{Leb}$ *is mixing and is the unique MME* *(???)*

Still get MME, but no proof of ergodicity/uniqueness/mixing

The Adler–Weiss–Bowen proof of uniqueness relies on ergodicity and the Gibbs property. Where to get ergodicity?

Theorem (C.–Knieper–War 2021, Adv. Math.)

For surfaces of genus ≥ 2 without conjugate points, a "coarse specification" argument establishes uniqueness of the MME.

With this in hand, Margulis argument (via Ricks) goes through.

Uniqueness using coarse specification

Theorem (C.–Thompson 2016, Adv. Math.)

Suppose a flow has specification ("uniform shadowing/gluing") at scale $\delta > 0$ and that obstructions to expansivity at scale $\epsilon > 40\delta$ have small entropy. Then there is a unique MME.

Surfaces of genus ≥ 2: expansivity condition holds for $\epsilon \leq \frac{1}{3} \operatorname{inj} M$

Specification holds at *any* scale in negative curvature

Morse Lemma: Fix 2 metrics on M: g_0 neg. curv., g no conj. pts. $\exists\ R > 0$ s.t. $\forall p, q \in X$, the g, g_0-geodesics $p \to q$ stay within R

$$g\text{-orbits} \xrightarrow{\text{Morse}} g_0\text{-orbits} \xrightarrow{\text{spec}} g_0\text{-orbit} \xrightarrow{\text{Morse}} g\text{-orbit}$$

δ-specification for g-geod. flow... but δ is probably too large!!

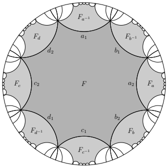
Salvation via residual finiteness

Expansivity scale is $\epsilon = \frac{1}{3}\,\mathrm{inj}\, M$.

Specification scale δ depends on R from Morse Lemma, likely large.

Get uniqueness if $\epsilon > 40\delta$, that is, $\mathrm{inj}\, M > 120\delta$. Probably false.

Solution: Replace M with a finite cover N with $\mathrm{inj}\, N$ big enough.



- Entropy-preserving bijection between flow-invariant measures on SM and SN.
- Theorem gives unique MME on SN
- Thus there is a unique MME on SM

Why possible? dim $M = 2$ implies $\pi_1(M)$ is *residually finite.*

Higher dimensions

Method works for higher-dim M with no conjugate points **if**

1. $\exists$ Riemannian metric g_0 on M with negative curvature;
2. divergence property: $c_1(0) = c_2(0) \Rightarrow d(c_1(t), c_2(t)) \to \infty$;
3. $\pi_1(M)$ is residually finite;
4. $\exists h^* < h_{\rm top}$ such that if μ-a.e. v has non-trivially overlapping horospheres, then $h_\mu \leq h^*$.

First is a real topological restriction: rules out Gromov example.

Second and third might be redundant? No example satisfying (1) where they are known to fail

Fourth is true if $\{v : H^s_v \cap H^u_v \text{ trivial}\}$ contains an open set. Unclear if this is always true.