# Unique equilibrium states in non-uniform settings

Vaughn Climenhaga

University of Toronto

October 22, 2011

Joint work with Daniel J. Thompson (Penn State)

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# The talk in one slide

specification $\Rightarrow$ intrinsic ergodicity (unique MME)

**Theorem (C.–Thompson)**

*If the complete obstruction to specification has small entropy, then the MME is unique.*

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specification + Bowen property $\Rightarrow$ unique equilibrium state

**Theorem (C.–Thompson)**

*If the complete obstruction to specification and the Bowen property has small pressure, then the equilibrium state is unique.*

# Topological pressure

Topological dynamical system:

- $X$ a compact metric space, $f\colon X \to X$ continuous
- $\mathcal{M} = \{$Borel $f$-invariant probability measures on $X\}$

Variational principle 1: $h_{\text{top}}(X, f) = \sup_{\mu\in\mathcal{M}} h_\mu(f)$

Variational principle 2: $P(\varphi) = \sup_{\mu\in\mathcal{M}} \left(h_\mu(f) + \int \varphi \, d\mu\right)$

Maximum achieved by MME/equilibrium state

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## Example

$$X = \Sigma_2^+ = \{0,1\}^{\mathbb{N}} \qquad p, q \in \mathbb{R} \qquad \varphi(x) = p\mathbf{1}_{[0]} + q\mathbf{1}_{[1]}$$

Then $P(\varphi) = \log(e^p + e^q)$ and the unique equilibrium state is $(\alpha, 1-\alpha)$-Bernoulli, where $\alpha = \frac{e^p}{e^p + e^q}$.

When is there a unique equilibrium state?

# Thermodynamics for shift spaces

Focus on shift spaces (subshifts):

- $X \subset \Sigma_p^+$ closed and $\sigma$-invariant
- language of $X$: $\mathcal{L} = \mathcal{L}(X) = \{x_1 \cdots x_n \mid x \in X, n \geq 1\}$
- words of length $n$: $\mathcal{L}_n = \{w \in \mathcal{L} \mid |w| = n\}$
- entropy: $h_{\mathrm{top}}(X, \sigma) = \lim_{n\to\infty} \frac{1}{n} \log \#\mathcal{L}_n$
- pressure: $P(\varphi) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{w\in\mathcal{L}_n} e^{\sup_{x\in[w]} S_n\varphi(x)}$
- $S_n\varphi(x) = \varphi(x) + \varphi(\sigma x) + \cdots + \varphi(\sigma^{n-1}x)$

$(X, \sigma)$ expansive $\Rightarrow$ entropy map $\mathcal{M} \to \mathbb{R}$ upper semi-continuous

$\Rightarrow$ equilibrium state exists

For shift spaces, the real question is uniqueness.

# Counterexample to uniqueness

**Example**

Let $X \subset \Sigma_5^+ = \{0, 1, 2, 1, 2\}^{\mathbb{N}}$ be the shift whose language $\mathcal{L}$ is defined by the following collection of forbidden words:

$$\{v0^n w,\ \ w0^n v \mid n < 2\max(|v|, |w|)\}.$$

- $(X, \sigma)$ is topologically transitive (indeed, mixing)
- $h_{\text{top}}(X, \sigma) = \log 2$
- 2 measures of maximal entropy:

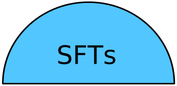
$$\nu = (\tfrac{1}{2}, \tfrac{1}{2})\text{-Bernoulli on } \{1, 2\}^{\mathbb{N}},$$
$$\mu = (\tfrac{1}{2}, \tfrac{1}{2})\text{-Bernoulli on } \{1, 2\}^{\mathbb{N}}.$$

Uniqueness of an MME can fail for transitive shifts.

# Classes of intrinsically ergodic shifts

The following are intrinsically ergodic (unique MME):

- Irreducible subshifts of finite type (Parry 1964)



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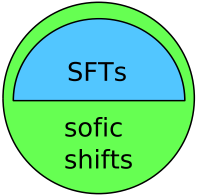

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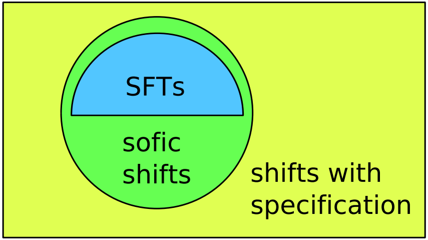

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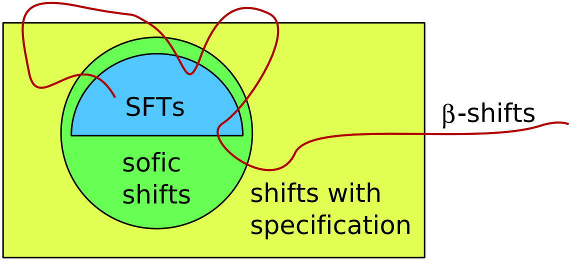
- $\beta$-shifts (Walters 1978, Hofbauer 1979)



# The motivating question

Intrinsic ergodicity is not necessarily preserved by factors.

- $X \subset \{0, 1, 2, 1, 2\}^{\mathbb{N}}$ as before
- $Y \subset \Sigma_6^+ = \{0, 1, 2, 1, 2, 3\}^{\mathbb{N}}$ by similar rule
- $X$ is a factor of $Y$; $Y$ is intrinsically ergodic; $X$ is not

Specification is preserved by factors, so intrinsic ergodicity survives.

What about $\beta$-shifts? (Klaus Thomsen, Mike Boyle)

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What about $\beta$-shifts? (Klaus Thomsen, Mike Boyle)

**Theorem (C.–Thompson 2010)**

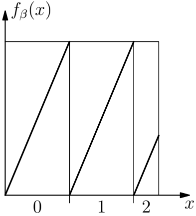
*$(X, \sigma)$ a subshift factor of a $\beta$-shift*
*$\Rightarrow$ complete obstruction to specification has zero entropy*
*$\Rightarrow$ $(X, \sigma)$ is intrinsically ergodic*

# $\beta$-shifts

For $\beta > 1$, $\Sigma_\beta$ is the coding space for the map

$$f_\beta \colon [0,1] \to [0,1], \qquad x \mapsto \beta x \pmod 1)$$

$1_\beta = a_1 a_2 \cdots$, where $1 = \sum_{n=1}^\infty a_n \beta^{-n}$



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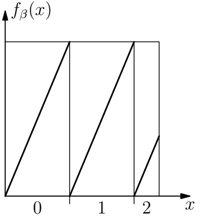


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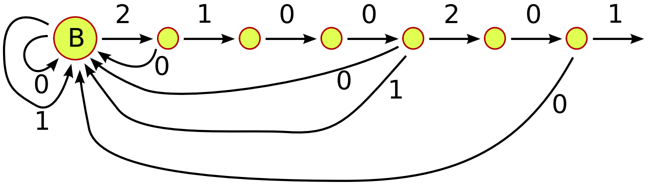
**Fact:** $x \in \Sigma_\beta$ iff $x$ labels a walk starting at **B** on the graph shown.
(Here $1_\beta = 2100201\dots$)



# Specification

Topological transitivity $\Rightarrow$ for every $w_1, \dots, w_m \in \mathcal{L}$ $\exists$ $z_i \in \mathcal{L}$ for which the concatenated word $w_1 z_1 w_2 z_2 \cdots z_{m-1} w_m$ is in $\mathcal{L}$.

**Definition**

$X$ has specification if $\exists$ $t \in \mathbb{N}$ such that $z_i$ can always be chosen to have length $t$, independently of $w_i$.

- Mixing SFTs and sofic shifts have specification.
- $\Sigma_\beta$ does not have specification if $1_\beta$ contains arbitrarily long strings of 0's. (Happens for Leb-a.e. $\beta > 1$.)

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The only obstruction to specification is the tail of the sequence $1_\beta$.

**Key idea: zero entropy obstructions are invisible to MMEs**

# Complete obstructions to specification

What do we mean by "obstruction"?

**Definition**

$\mathcal{G} \subset \mathcal{L}$ is a core for specification if

- Every $\mathcal{G}(M) := \{vw \in \mathcal{L} \mid v \in \mathcal{G}, |w| \leq M\}$ has specification

$\mathcal{C} \subset \mathcal{L}$ is a complete obstruction to specification if $\exists$ core $\mathcal{G}$ s.t.

- $\mathcal{L} = \mathcal{G}\mathcal{C} := \{vw \in \mathcal{L} \mid v \in \mathcal{G}, w \in \mathcal{C}\}$

We can glue words (orbit segments) together, provided we are allowed to remove an obstructing piece from the end of each word.

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**Example**

For the $\beta$-shift, take $\mathcal{C} = \{\text{prefixes of } 1_\beta\}$. ▸ GRAPH

- $\mathcal{C}$ = words whose path never returns to **B** (cusp excursions)
- $\mathcal{G}$ = words whose path begins and ends at **B**

# Small obstructions

**Theorem (C.–Thompson 2010)**

*If $X$ is a shift space, $\mathcal{C}$ is a complete obstruction to specification, and $h(\mathcal{C}) < h_{\mathrm{top}}(X, \sigma)$, then $(X, \sigma)$ is intrinsically ergodic.*

Remark: If the shadowing orbits can be taken to be periodic, then the unique MME is the limit of the measures $\mu_n = \delta_{\mathrm{Per}(n)}$.

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**Proposition**

$\pi : X \to \tilde{X}$ *a factor map,* $\mathcal{C} \subset \mathcal{L}(X)$ *a complete obstruction* $\Rightarrow$

- $\pi(\mathcal{C}) \subset \mathcal{L}(\tilde{X})$ *is also a complete obstruction*
- $h(\pi(\mathcal{C})) \leq h(\mathcal{C})$

# Coded systems

A shift space $X$ is coded if its language $\mathcal{L}$ is freely generated by a countable set of generators $\{w_n\}_{n\in\mathbb{N}} \subset \mathcal{L}$.

$$\mathcal{L} = \{\text{all subwords of } w_{n_1} w_{n_2} \cdots w_{n_k} \mid n_i \in \mathbb{N}\}$$

Canonical decomposition $\mathcal{L} = \mathcal{C}^p \mathcal{G} \mathcal{C}^s$ s.t. $\mathcal{G}(M)$ has specification:

$$\begin{aligned} \mathcal{G} &= \{w_{n_1} w_{n_2} \cdots w_{n_k} \mid n_i \in \mathbb{N}\} \\ \mathcal{C}^p &= \{\text{suffixes of } w_n \mid n \in \mathbb{N}\} \\ \mathcal{C}^s &= \{\text{prefixes of } w_n \mid n \in \mathbb{N}\} \end{aligned}$$

Let $\hat{h} = h(\{\text{prefixes and suffixes of generators}\})$.

- $\hat{h} < h_{\text{top}}(X, \sigma) \Rightarrow (X, \sigma)$ is intrinsically ergodic
- $\hat{h} = 0 \Rightarrow$ every subshift factor of $(X, \sigma)$ is intrinsically ergodic

# The Bowen property

Unique equilibrium states? Specification is not enough

## Example (Hofbauer 1977)

- $X = \Sigma_2^+ = \{0,1\}^{\mathbb{N}}$, $\varphi = \sum_k \mathbf{1}_{[0^k 1]} a_k$
- $a_k \nearrow 0$, $s_k = \sum_1^k a_i$, suppose $\sum e^{s_k} = 1$ and $\sum k e^{s_k} < \infty$

$\varphi$ has two equilibrium states: one fully supported, one atomic.

Given $X \subset \Sigma_p^+$ and $\mathcal{D} \subset \mathcal{L}(X)$, the $n$th variation of $\phi$ on $\mathcal{D}$ is

$$V_n(\mathcal{D}, \phi) = \sup_{w \in \mathcal{D}_n} \sup_{x,y \in [w]} |\phi(x) - \phi(y)|$$

## Definition

$\varphi$ has the Bowen property if $\sup_n V_n(\mathcal{L}, S_n\varphi) < \infty$.

Hölder continuous functions have the Bowen property.

# Unique equilibrium states

**Theorem (Bowen 1974)**

*Consider a system $(X, f)$ and a potential $\varphi \colon X \to \mathbb{R}$. Suppose*

1. *$X$ a compact metric space;*
2. *$f$ a continuous map;*
3. *$f$ is expansive;*
4. *$f$ has specification;*
5. *$\varphi$ has the Bowen property.*

*Then $\varphi$ has a unique equilibrium state.*

Goal: Replace these with non-uniform versions. Expect to get same result provided complete obstruction to all properties is small.

# Uniqueness in the presence of obstructions

## Definition

$\varphi$ has the Bowen property on $\mathcal{G}$ if

- $\sup_n V_n(\mathcal{G}_n, S_n\varphi) < \infty$

$\mathcal{C} \subset \mathcal{L}$ is a complete obstruction to the Bowen property for $\varphi$ if

- $\varphi$ is Bowen on some $\mathcal{G} \subset \mathcal{L}$ with $\mathcal{L} = \mathcal{G}\mathcal{C}$.

## Theorem (C.–Thompson 2011)

*Let $X$ be a shift space and $\varphi \in C(X)$. If $\exists$ a complete obstruction $\mathcal{C}$ to specification and the Bowen property for $\varphi$ that satisfies $P(\mathcal{C}, \varphi) < P(X, \varphi)$, then there is a unique equilibrium state for $\varphi$.*

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Remarks:

- $\mu_\varphi$ has a weak Gibbs property.
- If $\mathcal{G}(M)$ has (Per)-specification, then $\mu_\varphi = \lim_n \delta^{\varphi}_{\mathrm{Per}(n)}$.

## Equilibrium states for $\beta$-shifts

Let $(X, f) = (\Sigma_\beta, \sigma)$ be a $\beta$-shift.

**Theorem (Walters 1978)**

*Every Lipschitz potential $\varphi$ has a unique equilibrium state.*

**Theorem (Hofbauer–Keller 1982)**

*If $\varphi$ has the Bowen property and $\sup \varphi - \inf \varphi < h_{\mathrm{top}}(X, f)$, then $\varphi$ has a unique equilibrium state.*

**Theorem (C.–Thompson 2011)**

*Every Bowen potential $\varphi$ has a unique equilibrium state.*

# Variants of Manneville–Pomeau

**Example (Manneville–Pomeau map)**

- $X = [0,1]$, $\varepsilon \in (0,1)$, $f(x) = x + x^{1+\varepsilon} \pmod 1$
- $\varphi_t(x) = -t \log f'(x)$

$t = 1$: Two equilibrium states. $t < 1$: Unique equilibrium state.

**Example (Generalisation of Manneville–Pomeau)**

- $\gamma > 0 \quad \rightsquigarrow \quad f(x) = x + \gamma x^{1+\varepsilon} \pmod 1$.
- Topologically (semi-)conjugate to $\Sigma_\beta$ for some $\beta > 1$.
- For most values of $\gamma$, does not have specification.
- $\varphi_t(x) = -t \log f'(x)$ does not have the Bowen property.

**Theorem (C.–Thompson 2011)**

*For $t < 1$, there is a unique equilibrium state for $\varphi_t$.*

# Proof sketch (Bowen's proof)

Step 1. Constructible MME $\mu$

Step 2. $\mu$ is ergodic and Gibbs.

Step 3. No room for another MME $\nu \perp \mu$.

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# Proof sketch (Bowen's proof)

Step 0. Counting estimates. $Ce^{nh} \leq \#\mathcal{D}_n \leq C'e^{nh}$ whenever $\nu(\mathcal{D}_n) \gg 0$ for some MME $\nu$

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$$\mu((\mathcal{D} \cap \mathcal{G}(M))_n) \geq K(M)e^{-nh}\#(\mathcal{D} \cap \mathcal{G}(M))_n \geq K(M)C(M) > 0$$