Thermodynamics for non-uniformly mixing systems: factors of β-shifts are intrinsically ergodic

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January 19, 2011

Joint work with Daniel Thompson

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1. Introduction
   - Intrinsic ergodicity
   - Classical results
2. Motivating problem and solution
   - β-shifts
   - Intrinsic ergodicity for factors
3. General result
   - Specification and CGC-decompositions
   - A criterion for intrinsic ergodicity that passes to factors
   - Other examples and idea of proof

Basic thermodynamic concepts

Topological dynamical system:

- X a compact metric space, $f\colon X \to X$ continuous
- $\mathcal{M} = \{\text{Borel } f\text{-invariant probability measures on } X\}$

Variational principle: $h_{\text{top}}(X, f) = \sup_{\mu \in \mathcal{M}} h_\mu(f)$

- If $h_\mu(f) = h_{\text{top}}(X, f)$, then μ is a measure of maximal entropy (MME)
- (X, f) is intrinsically ergodic if there exists a unique MME

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Example: The full shift $\Sigma_p = \{1, \dots, p\}^{\mathbb{Z}}$ is intrinsically ergodic. The unique MME is $(\frac{1}{p}, \cdots, \frac{1}{p})$-Bernoulli measure.

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When is a transitive dynamical system intrinsically ergodic?

Motivation and context

More general variational principle for topological pressure $P(\varphi)$ of a continuous potential function $\varphi\colon X \to \mathbb{R}$

$$P(\varphi) = \sup_{\mu \in \mathcal{M}} \left(h_\mu(f) + \int \varphi \, d\mu \right)$$

If $h_\mu(f) + \int \varphi \, d\mu = P(\varphi)$, then μ is an equilibrium state.

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- Existence of a unique equilibrium state is connected to statistical properties, large deviations, multifractal analysis, phase transitions, etc.
- $\varphi \equiv 0$: reduces to intrinsic ergodicity. Techniques for showing intrinsic ergodicity usually generalise to help prove other thermodynamic results.

Intrinsic ergodicity for shift spaces

Focus on shift spaces (subshifts):

- $X \subset \Sigma_p$ or $X \subset \Sigma_p^+$, X closed and σ-invariant
- $\mathcal{L} = \mathcal{L}(X) = \{x_1 \cdots x_n \mid x \in X, n \geq 1\}$ is the language of X

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When is a transitive shift space intrinsically ergodic? *Not always.*

Example: $X \subset \Sigma_5 = \{0, 1, 2, 1, 2\}^{\mathbb{Z}}$. Define the language $\mathcal{L}$ by $v0^n w$, $w0^n v \in \mathcal{L}$ if and only if $n \geq 2\max(|v|, |w|)$.

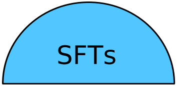
- (X, σ) is topologically transitive
- $h_{\text{top}}(X, \sigma) = \log 2$
- 2 measures of maximal entropy:

$$\nu = (\tfrac{1}{2}, \tfrac{1}{2})\text{-Bernoulli on } \{1, 2\}^{\mathbb{Z}},$$
$$\mu = (\tfrac{1}{2}, \tfrac{1}{2})\text{-Bernoulli on } \{1, 2\}^{\mathbb{Z}}.$$

Classes of intrinsically ergodic shifts

The following classes of shift spaces are intrinsically ergodic:

- Irreducible subshifts of finite type (Parry 1964)



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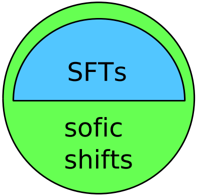

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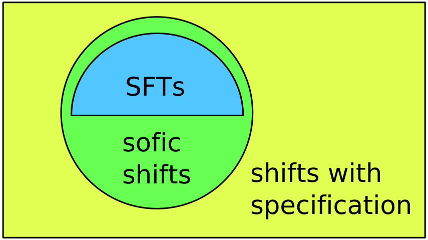

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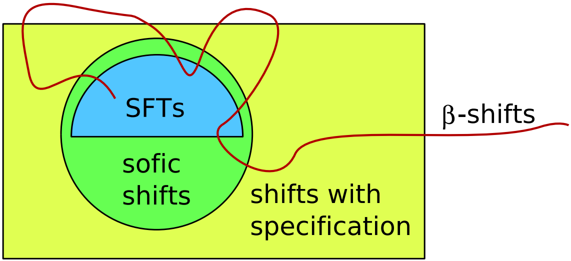
- β-shifts (Walters 1978, Hofbauer 1979)

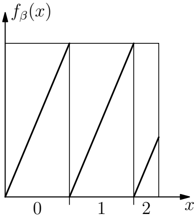


β-shifts

$\beta > 1$, $b = \lceil \beta \rceil$. The β-shift $\Sigma_\beta \subset \Sigma_b^+$ is the natural coding space for the map

$$f_\beta \colon [0,1] \to [0,1], \qquad x \mapsto \beta x \pmod 1$$

$1_\beta = a_1 a_2 \cdots$, where $1 = \sum_{n=1}^{\infty} a_n \beta^{-n}$



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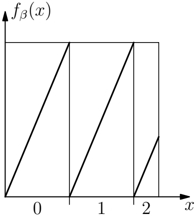

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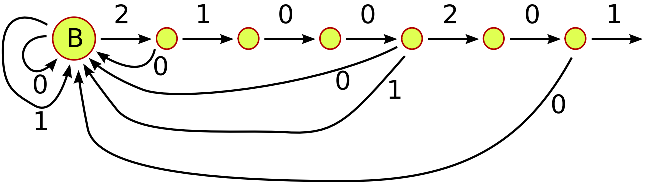
Fact: Sequences $x \in \Sigma_\beta$ are precisely those sequences in Σ_b that label trajectories of the following graph beginning at the vertex **B**.
(Here $1_\beta = 2100201\ldots$)

Σ_β $\mathcal{G}$ CGC



An open problem

Intrinsic ergodicity is not necessarily preserved by factors.

- $X \subset \{0, 1, 2, 1, 2\}^{\mathbb{Z}}$ as before
- $Y \subset \Sigma_6 = \{0, 1, 2, 1, 2, 3\}^{\mathbb{Z}}$ by similar rule
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What intrinsically ergodic classes are closed under factors?

- Closure of SFTs is class of sofic systems
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- Factors of β-shifts $=$?????

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Theorem (C.–Thompson 2010)

Yes.

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Theorem (C.–Thompson 2010)

Every subshift factor of a β-shift is intrinsically ergodic.

The classical specification property

- $\mathcal{L}$ = language for a shift space X
- $\mathcal{L} \leftrightarrow \{\text{cylinders in } X\}$
- $|w|$ = length of w, $\quad \mathcal{L}_n = \{w \in \mathcal{L} \mid |w| = n\}$

X has specification if there exists $t \in \mathbb{N}$ such that for every $w_1, \dots, w_m \in \mathcal{L}$, there exist $z_1, \dots, z_{m-1} \in \mathcal{L}_t$ for which the concatenated word $w_1 z_1 w_2 z_2 \cdots z_{m-1} w_m$ is in $\mathcal{L}$.

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Topological transitivity guarantees the existence of such words $z_i \in \mathcal{L}$. Specification demands that the words z_i can be chosen to have uniformly bounded length t, where t is independent of the words w_i and their lengths.

Shifts with and without specification

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▸ GRAPH

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GRAPH

Σ_β does not have specification for Lebesgue-a.e. $\beta > 1$.

We must replace specification with a property that

- holds for every β-shift;
- implies intrinsic ergodicity;
- is preserved by factors.

A restricted version of the specification property

Fix a subset $\mathcal{G} \subset \mathcal{L}$. We say that $\mathcal{G}$ has specification if there exists $t \in \mathbb{N}$ such that for every $w_1, \dots, w_m \in \mathcal{G}$, there exist $z_1, \dots, z_{m-1} \in \mathcal{L}_t$ for which the concatenated word $x := w_1 z_1 w_2 z_2 \cdots z_{m-1} w_m$ is in $\mathcal{L}$.

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Example: For $X = \Sigma_\beta$, let $\mathcal{G}$ be the set of words corresponding to paths that begin and end at **B**. Then $\mathcal{G}$ has (Per)-specification with $t = 0$.

GRAPH

Decomposing the language

A CGC-decomposition of the language $\mathcal{L}$ is a collection of words $\mathcal{C}^p, \mathcal{G}, \mathcal{C}^s \subset \mathcal{L}$ with the following properties.

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Example: For $X = \Sigma_\beta$, let $\mathcal{C}^p = \emptyset$ and let $\mathcal{C}^s$ be the set of words corresponding to paths that begin at **B** and never return. Then $(\mathcal{C}^p, \mathcal{G}, \mathcal{C}^s)$ is a uniform CGC-decomposition.

GRAPH

Intrinsic ergodicity for shifts with CGC-decompositions

Given a collection of words $\mathcal{D} \subset \mathcal{L}$, let $h(\mathcal{D}) = \overline{\lim}_{n\to\infty} \frac{1}{n} \log \#\mathcal{D}_n$.
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Theorem (C.–Thompson 2010)

Let X be a shift space admitting a uniform CGC-decomposition.

If $h(\mathcal{C}^p \cup \mathcal{C}^s) < h_{\text{top}}(X, \sigma)$, then (X, σ) is intrinsically ergodic.

If $\mathcal{G}$ has (Per)-specification, then the unique MME is the limit of the periodic orbit measures $\mu_n = \frac{1}{\#\{x | f^n(x) = x\}} \sum_{f^n(x)=x} \delta_x$.

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Example: For $X = \Sigma_\beta$, let $x = 1_\beta$. Then $(\mathcal{C}^p \cup \mathcal{C}^s)_n = \{x_1 \cdots x_n\}$, and so $h(\mathcal{C}^p \cup \mathcal{C}^s) = 0$. Thus (Σ_β, σ) is intrinsically ergodic.

Behaviour under factors

Let $(\tilde{X}, \sigma)$ be a factor of (X, σ), and let $\mathcal{L}, \tilde{\mathcal{L}}$ be the languages.

- If $\mathcal{L}$ has a uniform CGC-decomposition, then so does $\tilde{\mathcal{L}}$. Futhermore, $h(\tilde{\mathcal{C}}^p \cup \tilde{\mathcal{C}}^s) \leq h(\mathcal{C}^p \cup \mathcal{C}^s)$.

Every factor with $h_{\mathrm{top}}(\tilde{X}, \sigma) > h(\mathcal{C}^p \cup \mathcal{C}^s)$ is intrinsically ergodic.

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Dichotomy for shifts with uniform CGC-decompositions:

Either $h_{\text{top}}(X, \sigma) > 0$, or X comprises a single periodic orbit.

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Theorem (C.–Thompson 2010)

Let X be a shift space admitting a uniform CGC-decomposition.

If $h(\mathcal{C}^p \cup \mathcal{C}^s) = 0$, then every subshift factor of (X, σ) is intrinsically ergodic.

S-gap shifts

Fix $S \subset \mathbb{N}$ and suppose S is infinite. The associated S-gap shift is the subshift $\Sigma_S \subset \{0,1\}^{\mathbb{Z}}$ with language

$$\mathcal{L} = \{0^k 1 0^{n_1} 1 0^{n_2} 1 \cdots 1 0^{n_j} 1 0^{\ell} \mid n_i \in S, k, \ell \in \mathbb{N}\}.$$

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A uniform CGC-decomposition for Σ_S is given by

$$\begin{aligned} \mathcal{G} &= \{0^n 1 \mid n \in S\} \\ \mathcal{C}^p &= \{0^k 1 \mid k \geq 0\} \\ \mathcal{C}^s &= \{0^\ell \mid \ell \geq 1\} \end{aligned}$$

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Then $\#(\mathcal{C}^p \cup \mathcal{C}^s)_n = 2$ for all $n \geq 1$, and so $h(\mathcal{C}^p \cup \mathcal{C}^s) = 0$. It follows that every subshift factor of an S-gap shift is intrinsically ergodic.

Coded systems

A shift space X is coded if its language $\mathcal{L}$ is freely generated by a countable set of generators $\{w_n\}_{n\in\mathbb{N}} \subset \mathcal{L}$.

$$\mathcal{L} = \{\text{all subwords of } w_{n_1} w_{n_2} \cdots w_{n_k} \mid n_i \in \mathbb{N}\}$$

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Let $\hat{h} = h(\{\text{prefixes and suffixes of generators}\})$.

- $\hat{h} < h_{\text{top}}(X,\sigma) \Rightarrow (X,\sigma)$ is intrinsically ergodic
- $\hat{h} = 0 \Rightarrow$ every subshift factor of (X,σ) is intrinsically ergodic

Structure of proof

Step 1. Estimates on number of words of a given length

$$\#\mathcal{L}_{m+n} \leq (\#\mathcal{L}_m)(\#\mathcal{L}_n) \Rightarrow \#\mathcal{L}_n \geq e^{nh}$$
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Step 2. Build a measure of maximal entropy as a limit of δ-measures on (n, ε)-separated orbits. Use counting estimates to obtain a Gibbs property

$$w_1 \cdots w_n \in \mathcal{G}(M) \Rightarrow \mu([w_1 \cdots w_n]) \geq K_M e^{-nh}$$

Structure of proof

Step 3. The mme μ is ergodic

$$\mu(P) > 0, \mu(Q) > 0 \Rightarrow \overline{\lim_{n\to\infty}} \mu(P \cap \sigma^{-n}(Q)) > 0$$

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Choose $\mathcal{D}$ such that $\nu(\mathcal{D}_n) \to 1$ and $\mu(\mathcal{D}_n) \to 0$.

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The Gibbs property says that

$$\mu(\mathcal{D}_n) \geq K_M e^{-nh} \#(\mathcal{D}_n \cap \mathcal{G}(M)) \geq K_M C > 0$$

Thus no such ν can exist, and μ is unique.