THE BIGNESS OF THINGS

PMASS colloquium, March 17, 2011 (Thursday)

How big is it? Of course the answer depends on "it", but so does the meaning of "big"... and context is crucial.

How big is a... crowd? → # of people, unless ~~they're~~ on an elevator, in which case...

→ total weight

fish? → length (for the photo)

→ weight (for supper)

city? → # of people (for the census)

→ area (for the planning council)

→ diameter (for the road trip)

house? → floor area (to carpet it)

→ volume (to heat it)

→ # of bedrooms (to live in it)

Most familiar notions of "bignness" can be described as either

⓪ cardinality
① length
② area
③ volume

or "weighted" versions of these.
(weight = $\int$ density d(volume))

Consider subsets of $\mathbb{R}^3$. Which of these notions are best suited?

(a) finite set of points (0-dim)

(b) curve (1-dim)

(c) surface (2-dim)

(d) region with interior (3-dim)

(0) (b)-(d) have infinite cardinality, but it works for (a)

(1) (a) has zero length - why? - cover each pt with tiny intervals
(digression: $\mathbb{Q}$ has 0 "length" in $\mathbb{R}$)
(b) has finite positive length
(c)-(d) have infinite length - no curve of finite length can cover them
(possible objection: they do have finite diameter, which is a length - but a length <u>of a projection</u>)

(2) (a)-(b) have zero area, (d) has infinite... but (c) has finite positive
(mention Goldilocks phenomenon)

(3) (a)-(c) have zero volume, but (d) has finite positive volume

<u>Note</u> We haven't given real definitions of length, area, volume for arbitrary sets. This takes measure theory and leads to complications (Banach-Tarski, etc.) - we go a different direction.

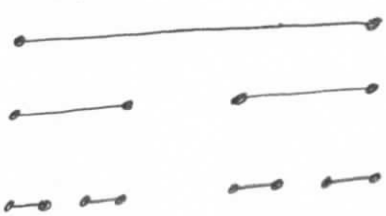
MORAL Before we can say how "big" something is, we need to know its "dimension". Indeed, dimension itself is a notion of bigness.

Example 1 <u>Cantor set</u>

$C_0 = [0,1]$

$C_1 = [0,\frac{1}{3}] \cup [\frac{2}{3},1]$

$C_2 = [0,\frac{1}{9}] \cup [\frac{2}{9},\frac{1}{3}] \cup [\frac{2}{3},\frac{7}{9}] \cup [\frac{8}{9},1]$



$C_n \to C_{n+1}$: remove middle third from each of 2^n intervals.

$\boxed{C = \bigcap_{n \geq 0} C_n}$

<u>Fact 1:</u> C is uncountable. (Bijection $\pi : \{0,1\}^{\mathbb{N}} \to C$)
(choose left branch on 0, right branch on 1)

<u>Fact 2:</u> C has zero length. ($\text{length}(C_n) = 2^n 3^{-n} \to 0$)

What's its dimension? <u>Between 0 and 1.</u>

$\boxed{\text{Example 2}}$ Koch curve

$K_0 =$ 

$K_1 =$ 

$K_n \to K_{n+1}$:
replace each line segment with a scaled-down copy of K_0

$$\boxed{K = \lim_{n\to\infty} K_n}$$

Fact 1: $\text{length}(K) = \infty$. $\quad (\text{length}(K_n) = 4^n 3^{-n})$

Fact 2: $\text{area}(K) = 0$. $\quad$ (exercise – cover with small rectangles)

What's its dimension? Between 1 and 2.

Back to generalities... what is dimension?

- # of parameters/coordinates, but this is always an integer
- scaling exponent:

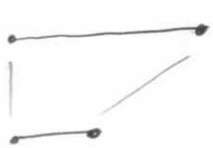
  $\lambda > 0, \quad E \subset \mathbb{R}^3, \quad \lambda E = \{\lambda \vec{x} \mid \vec{x} \in E\}$

$$\left.\begin{aligned}
\text{Volume}(\lambda E) &= \lambda^3 \,\text{Vol}(E) \\
\text{area}(\lambda E) &= \lambda^2 \,\text{area}(E) \\
\text{Length}(\lambda E) &= \lambda^1 \,\text{Length}(E) \\
\text{Cardinality}(\lambda E) &= \lambda^0 \,\text{Card}(E)
\end{aligned}\right\} \text{self-similarity}$$

Remark The "correct" thing to do now is look for measures
$\mu_\alpha : \{\text{subsets of } \mathbb{R}^3\} \to \mathbb{R}$ s.t. $\mu_\alpha(\lambda E) = \lambda^\alpha \mu_\alpha(E)$
These are the α-dimensional Hausdorff measures, but this would take us too far afield. Instead...

Look at self-similarity for sets E, with $\lambda = \frac{1}{2}$

$E = [0,1]$ $\qquad$ $E = [0,1]^2$ $\qquad$ $E = [0,1]^3$



Need 2 copies... $\qquad$ Need 4 copies... $\qquad$ Need 8 copies...

... to recover original shape.

MORAL — E has self-similarity if it is a union of n copies of λE. In this case $n = \lambda^{-\alpha}$, where α is the dimension of E.

NB This is not the real definition of Hausdorff dimension

Examples

* interval: $\lambda = \frac{1}{2}$, $n = 2 \Rightarrow \alpha = \frac{-\log n}{\log \lambda} = \frac{-\log 2}{\log 1/2} = 1$

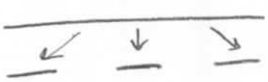
* square: $\lambda = \frac{1}{2}$, $n = 4 \Rightarrow \alpha = \frac{-\log 4}{\log 1/2} = 2$

* cube: $\lambda = \frac{1}{2}$, $n = 8 \Rightarrow \alpha = \frac{-\log 8}{\log 1/2} = 3$

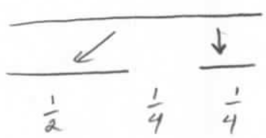
* Cantor set: $\lambda = \frac{1}{3}$, $n = 2 \Rightarrow \alpha = \frac{-\log 2}{\log 1/3} = \frac{\log 2}{\log 3} \in (0,1)$

* Koch curve: $\lambda = \frac{1}{3}$, $n = 4 \Rightarrow \alpha = \frac{\log 4}{\log 3} \in (1,2)$

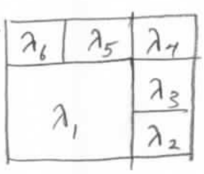
There are other Cantor sets:

① 

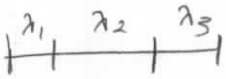
3 intervals of length $\frac{1}{5} \Rightarrow$ dimension $= \frac{\log 3}{\log 5}$

② 

What to do??? Think about interval, square, etc., with *different* λ's...



$$\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2 + \lambda_5^2 + \lambda_6^2 = 1$$



$$\lambda_1 + \lambda_2 + \lambda_3 = 1$$

$$\boxed{E = \bigcup \lambda_i E \longrightarrow \text{solve } \sum_i \lambda_i^\alpha = 1}$$

Moran's equation

For $(\frac{1}{2}, \frac{1}{4})$ - Cantor set, this gives

$$\left(\frac{1}{2}\right)^{\alpha} + \left(\frac{1}{4}\right)^{\alpha} = 1 \quad \leadsto \quad \begin{array}{l} x = 2^{-\alpha} \\ x^2 + x = 1 \end{array}$$

$$x = \frac{-1 \pm \sqrt{5}}{2} \quad \Rightarrow \quad 2^{-\alpha} = \frac{\sqrt{5}-1}{2}$$

$$\alpha = \frac{-\log\left(\frac{\sqrt{5}-1}{2}\right)}{\log 2} = \boxed{1 - \frac{\log(\sqrt{5}-1)}{\log 2}}$$

AND NOW, FOR SOMETHING (apparently) COMPLETELY DIFFERENT...

Stochastic process #1: Flip a coin repeatedly, write down H or T...

Stochastic process #2: Roll a die repeatedly, write down number (1-6)...

Which one is "bigger"? (More possibilities, more uncertainty...)
We think #2 is... but why?

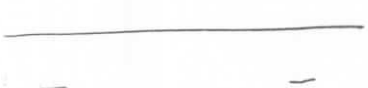
"Bigness" of process = amount of information to record after each iteration of the experiment.

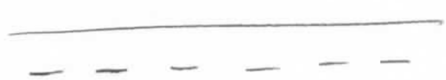
How to measure information? Number of bits it takes to store it.

eg. a single coin flip carries 1 bit of information, so Process #1 has a "bigness" of 1 bit. This is called entropy & denoted h.

In general, n options require $\log_2 n$ bits... so Process #2 has entropy $h = \log_2 6 > 1$... its bigger.

The entropies were $\frac{\log 2}{\log 2}$ and $\frac{\log 6}{\log 2}$ – compare this to two Cantor sets with 2 & 6 branches and $\lambda = \frac{1}{10}$:





$$\dim = \frac{\log 2}{\log 10} \qquad\qquad \dim = \frac{\log 6}{\log 10}$$

What if all possibilities are not equal? Say I paint 2 faces of the die red, and the ~~other~~ 4 blue. Then

$$P(\text{red}) = \tfrac{1}{3} \qquad P(\text{blue}) = \tfrac{2}{3}$$

There's less information here. (Imagine $\frac{1}{100}$ & $\frac{99}{100}$...)
But how much? Storage space is the same... (unless we compress)

[Def] Information content of an event $= -\log_2(\text{probability})$

entropy = <u>expected</u> information content of each experiment

So information (red) $= -\log_2\left(\frac{1}{3}\right) = \log_2 3$

information (blue) $= -\log_2\left(\frac{2}{3}\right) = \log_2\left(\frac{3}{2}\right)$

$$h = \frac{1}{3}\left(-\log_2\left(\frac{1}{3}\right)\right) + \frac{2}{3}\left(-\log_2\left(\frac{2}{3}\right)\right) = \frac{1}{3}\log_2 3 + \frac{2}{3}\log_2\left(\frac{3}{2}\right)$$

$$\boxed{h = (\log_2 3) - \frac{2}{3}}$$

ASIDE dimension of $\left(\frac{1}{2},\frac{1}{4}\right)$-Cantor from earlier is

$$\sup_{t\in[0,1]} \frac{-t\log t - (1-t)\log(1-t)}{t\log 2 + (1-t)\log 4}$$

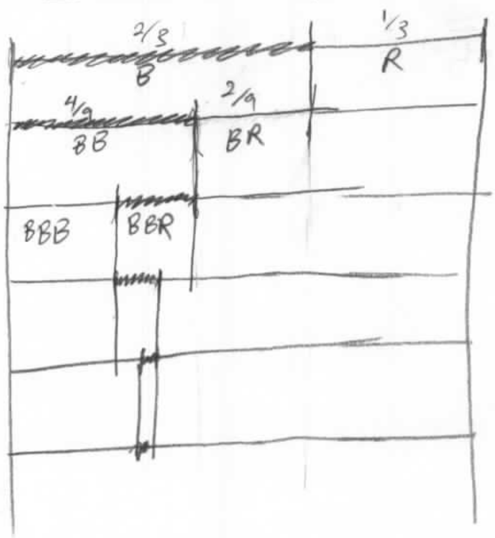
(Think of starting with 3 equally probable outcomes, so $h = \log_2 3$, then forgetting the difference between two of them, which lets you save 1 bit $\frac{2}{3}$ of the time.)

[Aside] Related to dimension: $\dim = h / \text{Lyapunov exponent}$

Final remark - <u>Shannon's source coding theorem</u>

n iterations of the experiment can have results stored in nh bits of information under optimum compression.

eg <u>Arithmetic coding</u> of B,R-die above. Suppose we roll BBRBRB.



Need to specify interval. Width of "average" interval after n experiments is $\left(\frac{1}{3}\right)^{\frac{1}{3}n}\left(\frac{2}{3}\right)^{\frac{2}{3}n}$

$= 2^{-nh}$, so it takes nh bytes to give that much precision.