Multifractals, dimension theory, and thermodynamics for general dynamical systems

X cpt met sp, $f: X \circlearrowleft$ cts, $\varphi: X \to \mathbb{R}$ an observable

$S_n\varphi(x) = \sum_{k=0}^{n-1} \varphi(f^k(x))$ Birkhoff sum – what do we know about the Birkhoff average $\frac{1}{n} S_n\varphi(x)$?

$\boxed{\text{ergodic theorem}}$ $\frac{1}{n} S_n \varphi(x) \to \int \varphi \, d\mu$ for μ-a.e. x if $\mu \in \mathcal{M}_e^f(X)$ $= \{f\text{-inv. erg. prob. meas.}\}$ DIFFERENT μ $\Rightarrow$ DIFFERENT AVERAGE

We expect $\mathcal{M}^f(X)$ to contain many inv. meas. Not clear which is most meaningful – so how do we understand asymptotics of $S_n\varphi$?

$\boxed{\text{Def'n}}$ $K_\alpha = \{x \in X \mid \frac{1}{n} S_n \varphi(x) \to \alpha\}$, $\alpha \in \mathbb{R}$ level set

What are the sets K_α like? (Every ergodic measure sits on one level set).

$\boxed{\text{Example}}$ $\sigma: \Sigma_2^+ \circlearrowleft$ is the full shift, $\varphi(x) = \mathbb{1}_{[0]}(x)$, so $\frac{1}{n} S_n \varphi(x)$ gives asymptotic frequency of 0's. Then K_α is dense for $0 \le \alpha \le 1$ and empty otherwise.

Intuition: K_0 is smaller than $K_{1/2}$. But in what sense?

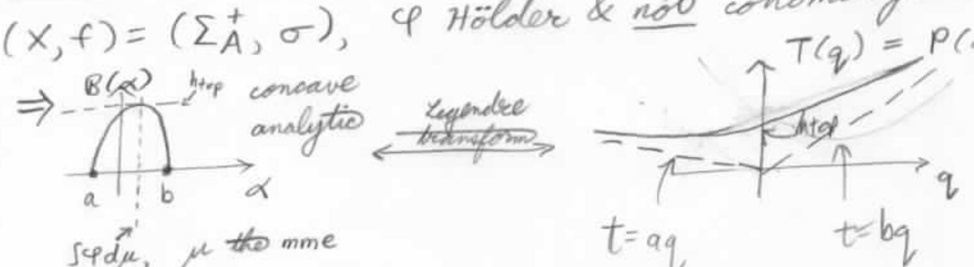
$\boxed{\text{Def'n}}$ $X = (\bigcup_{\alpha \in \mathbb{R}} K_\alpha) \cup X'$ is a <u>multifractal decomposition</u>, and the function $\mathcal{B}(\alpha) = h_{top}(K_\alpha)$ is the <u>Birkhoff spectrum</u>.

$\boxed{\text{Rmk}}$ Here h_{top} is a dimension: $\dim_H$ uses $B(x,r)$ & sends $r \to 0$, h_{top} uses $B(x,n,\delta)$ & sends $n \to \infty$. (Bowen's def'n)

$\boxed{\text{Question}}$ What does $\mathcal{B}(\alpha)$ look like?

$\boxed{\text{THEOREM}}$ (Pesin, Weiss 2001)

$(X,f) = (\Sigma_A^+, \sigma)$, φ Hölder & <u>not</u> cohomologous to a constant



$\boxed{\text{Def'n}}$ $\mathcal{B}(\alpha)$ concave $\Rightarrow \mathcal{B}^{L_1}(q) = \sup_{\alpha \in \mathbb{R}} (\mathcal{B}(\alpha) + q\alpha)$

$T(q)$ convex $\Rightarrow T^{L_2}(\alpha) = \inf_{q \in \mathbb{R}} (T(q) - q\alpha)$

If C^1, then $\mathcal{B}'(\alpha) = -q \Rightarrow \mathcal{B}^{L_1}(q) = \mathcal{B}(\alpha) + q\alpha$

$T'(q) = \alpha \Rightarrow T^{L_2}(\alpha) = T(q) - q\alpha$

$\boxed{\text{Pf of Thm}}$ PW use dimension spectrum for Gibbs measure for φ

* roundabout & indirect - uses lots of extra structure, not obvious how to generalise to ~~other settings~~

Idea for more direct pf / justification for Legendre transform:

$$(*) \quad T(q) = P_X(q\varphi) = P_{\cup_\alpha K_\alpha}(q\varphi) = \sup_\alpha P_{K_\alpha}(q\varphi) = \sup_\alpha (h_{top}(K_\alpha) + q\alpha) = \mathcal{B}^{L_1}(q)$$

$\boxed{\text{PROBLEMS}}$ ① neglects X' ② uncountably many α

③ also, no a priori guarantee that $\mathcal{B}$ concave.

Can fix ① & ② to establish $(*)$ in general. ③ requires more.

$\boxed{\text{THEOREM}}$ (C.)

X cpt, $f: X \circlearrowleft$ cts, $\varphi: X \to \mathbb{R}$ cts $\Rightarrow$ (I) $T = \mathcal{B}^{L_1}$ (II) $\alpha \notin [a,b] \Rightarrow K_\alpha = \emptyset$.

(III) If $q\varphi$ has an equilibrium state $\forall q \in (q_1, q_2)$ and T is C^1 on (q_1, q_2), then $\mathcal{B}(\alpha) = T^{L_2}(\alpha) \quad \forall \alpha \in (\alpha_1, \alpha_2) = T'(q_1, q_2)$

$\boxed{\text{Sketch Pf}}$ (I) $(T \leq \mathcal{B}^{L_1})$ $\quad T(q) = \sup_\mu (h(\mu) + q\int \varphi\, d\mu) \leq \sup_\alpha (\mathcal{B}(\alpha) + q\alpha) = \mathcal{B}^{L_1}(q)$

$(T \geq \mathcal{B}^{L_1})$ $\quad \forall \varepsilon > 0, \; K_\alpha \subset \bigcup_N F_\alpha^{\varepsilon, N} = \bigcup_N \{x \in X \mid |\tfrac{1}{n} S_n\varphi(x) - \alpha| \leq \varepsilon \;\; \forall n \geq N\}$

$\therefore \; \mathcal{B}(\alpha) \leq \sup_N h_{top} F_\alpha^{\varepsilon,N} \leq \sup_N \underline{Ch}_{top} F_\alpha^{\varepsilon,N}$

$\forall N$, build a measure μ with $|\int \varphi\, d\mu - \alpha| \leq \varepsilon$, $\; h(\mu) \geq \underline{Ch}_{top} F_\alpha^{\varepsilon, N}$

$\therefore \; \mathcal{B}(\alpha) + q\alpha - \varepsilon \leq T(q)$, and α, ε, q were arbitrary.

(II) $x \in K_\alpha$, $\; \mu_n = \frac{1}{n}\sum_{k=0}^{n-1} \delta_{f^k(x)}$, $\; \mu_{n_j} \to \mu \Rightarrow T(q) \geq \int q\varphi\, d\mu = q\alpha \;\; \forall q$

(III) μ equilibrium for $q\varphi \Rightarrow D^- T(q) \leq \int q\varphi\, d\mu \leq D^+ T(q)$

C^1 + intermediate value theorem $\Rightarrow$ get ergodic eq. state μ with $\mu(K_\alpha) = 1$

Then $\mathcal{B}(\alpha) \geq h(\mu) = P(q\varphi) - q\alpha = T(q) - q\alpha \geq T^{L_2}(\alpha)$ ∎

$\boxed{\text{Rmk}}$ If $\mu \mapsto h(\mu)$ upper semi-cts, then a weak* limit of equilibrium states is again an eq st. In this case, if $\exists!$ eq st. for $q\varphi$ on (q_1, q_2), then T is C^1 (take left & right limits).

* Example due to Varandas, Viana - first multifractal results

* For expansive maps with (almost) specification, can also use results of Pfister & Sullivan. (Takens & Verbitskiy, Thompson).

$\boxed{\text{Question}}$ What about discts potentials? Continuity only used in construction of μ with $\int \varphi\, d\mu \approx \alpha$, $h(\mu) \gtrsim \mathcal{B}(\alpha)$. If φ discts, let $C(\varphi)$ be the set of discontinuities, and suppose φ bdd above & below.

$\boxed{\text{THEOREM}}$ If $\mu(\overline{C(\varphi)})=0 \ \forall \mu \in \mathcal{M}^f(X)$, then same result holds.

More generally, let $I_A = \{\alpha \in \mathbb{R} \mid T^{L_2}(\alpha) > h_0\}$, $h_0 = \underline{Ch}_{top}(C(\varphi))$, and $I_Q = \{q \in \mathbb{R} \mid T$ has subtangent at q with y-intercept $> h_0\}$. Then

(I) $T(q) = \sup_{\alpha \in I_A} (\mathcal{B}(\alpha) + q\alpha) \quad \forall q \in I_Q$

(II) $\alpha \notin I_A \Rightarrow \mathcal{B}(\alpha) \leq h_0$

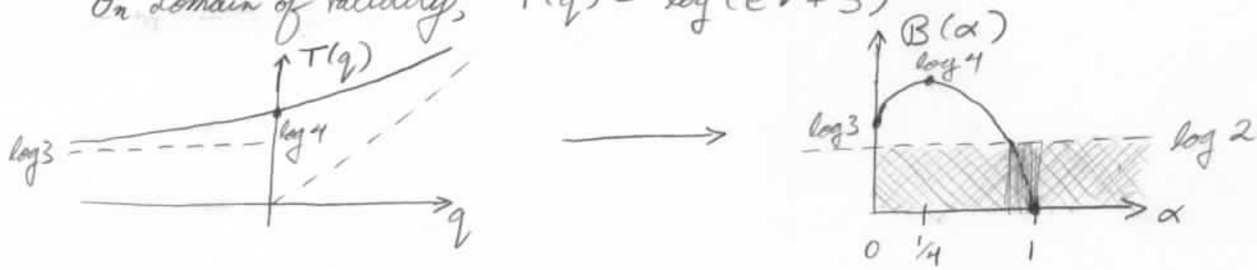
(III) existence $+ C^1$ on $(q_1, q_2) \Rightarrow \mathcal{B} = T^{L_2}$ on $(\alpha_1, \alpha_2) \cap I_A$.

$\boxed{\text{Example}}$ $f = $ ~~tent map~~, $\varphi = \mathbb{1}_{[0,\frac{1}{2}]}$

In higher dimensions, can consider $\varphi = \mathbb{1}_Z$ where ∂Z "large".

$f: \mathbb{T}^2 \circlearrowleft$, $x \mapsto 2x$, $\varphi = \mathbb{1}_Z$, $Z = [0,\frac{1}{2}]^2 \Rightarrow \underline{Ch}_{top}(\partial Z) = \log 2$

On domain of validity, $T(q) = \log(e^q + 3)$



$\boxed{\text{OTHER SPECTRA}}$ local asymptotic quantity + global dimension

$h_\nu(x) + h_{top} \Rightarrow$ entropy spectrum (follows for Gibbs)

$d_\nu(x) + \dim_H \Rightarrow$ dimension spectrum

How to address this in the above framework? If f is conformal without critical points, then the following hold:

① $d_\nu(x) = \dfrac{h_\nu(x)}{\lambda(x)} = \lim_{n\to\infty} \dfrac{-S_n\varphi(x)}{S_n \log|f'(x)|}$ (ν Gibbs for φ)

② $\dim_H Z = t^* \Leftrightarrow t^* = \inf\{t \in \mathbb{R} \mid P_Z(-t\log|f'|) = 0\}$

(need only a bounded/tempered contraction condition)

Write $\psi = \log|f'|$, then this reduces to

$$K_\alpha = \left\{x \in X \;\middle|\; \frac{-S_n\varphi(x)}{S_n\psi(x)} \to \alpha\right\}$$

$$\mathcal{D}(\alpha) = \dim_\psi K_\alpha = \inf\{t \in \mathbb{R} \mid P_{K_\alpha}(-t\psi) = 0\}$$

(so everything is back in terms of dynamically-defined quantities)

Idea is still to find an equilibrium state that sits on K_α.
If $\mu(K_\alpha)=1$ and $\int \psi \, d\mu = \lambda$ for some μ, μ an equilibrium state for some ~~potential~~ ζ at which P is differentiable, then

$$\frac{d}{dt} P(\zeta + t\psi)\Big|_{t=0} = \lambda, \quad \frac{d}{dq} P(\zeta + q\varphi)\Big|_{q=0} = -\alpha\lambda$$

Want to replicate previous proof by finding a single function $T(q)$ with slope $\pm\alpha$, so define it implicitly:

$$P(q\varphi - T(q)\psi) = 0$$

(more precisely, do $T(q) = \inf\{t \in \mathbb{R} \mid P(q\varphi - t\psi) = 0\}$)

Then if μ is an eq. st. for $q\varphi - T(q)\psi$, get

$$0 = \frac{d}{dq} P(q\varphi - T(q)\psi) = \int \varphi \, d\mu - T'(q) \int \psi \, d\mu \Rightarrow T'(q) = \frac{\int \varphi \, d\mu}{\int \psi \, d\mu}$$

$\therefore \mu(K_\alpha) = 1$ for $\alpha = -T'(q)$

Furthermore, $h(\mu) + q\int \varphi \, d\mu - T(q)\int \psi \, d\mu = 0$

$$\Rightarrow \frac{h(\mu)}{\int \psi \, d\mu} = T(q) - q\frac{\int \varphi \, d\mu}{\int \psi \, d\mu}$$

$\Rightarrow \mathcal{D}(\alpha) \geq \dim_\psi \mu = T(q) + q\alpha$

This replicates the proof of (III) – can do (I) & (II) as well (need assumptions on eventual positivity of $S_n\psi$).

$\boxed{\text{Final remark}}$ Points of non-differentiability of T → phase transitions. These can be dealt with via approximation from within: $\{X_n\}$ cpt, inv. s.t.

① $q \mapsto P_{X_n}(q\varphi)$ smooth

② $P_{X_n}(q\varphi) \to P_X(q\varphi)$