A non-uniform Bowen's equation and connections to multifractal analysis

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Global dimensional characteristics

Quantify a fractal set Z using various global dimensional quantities.

Hausdorff dimension $\dim_H Z$: critical value of t for t-dimensional Hausdorff measure $m(Z,t)$, defined using covers by metric balls $B(x,\varepsilon)$.

Replace $B(x,\varepsilon)$ with Bowen balls $B(x,n,\delta)$: obtain topological entropy $h_{\text{top}} Z$ as a dimensional characteristic (Bowen).

Give the balls $B(x,n,\delta)$ different weights according to $S_n\varphi(x)$ (Birkhoff sum): obtain topological pressure $P_Z(\varphi)$ as a dimensional characteristic (Pesin–Pitskel').

(The latter two quantities depend on the underlying system f.)

Local dimensional characteristics

Given an invariant measure μ for f, we have analogous local quantities: pointwise dimension d_μ, local entropy h_μ, Lyapunov exponent λ.

$$d_\mu(x) = \lim_{\varepsilon\to 0} \frac{\log \mu(B(x,\varepsilon))}{\log \varepsilon} \qquad \mu(B(x,\varepsilon)) \approx \varepsilon^{d_\mu(x)}$$

$$h_\mu(x) = \lim_{\delta\to 0} \varliminf_{n\to\infty} -\frac{1}{n}\mu(B(x,n,\delta)) \qquad \mu(B(x,n,\delta)) \approx e^{-nh_\mu(x)}$$

$$\lambda(x) = \lim_{n\to\infty} \frac{1}{n} \log \left|(f^n)'(x)\right| \qquad B(x,n,\delta) \approx B(x,\delta e^{-n\lambda(x)})$$

Where the limits exist, we expect to find

$$d_\mu(x) = \lim_{\varepsilon\to 0} \frac{\log \mu(B(x,\varepsilon))}{\log \varepsilon} = \lim_{n\to\infty} \frac{\log \mu(B(x,n,\delta))}{-n\lambda(x)} = \frac{h_\mu(x)}{\lambda(x)}$$

From local to global

Local quantities can be used to determine global quantities.

- If $\mu(Z) > 0$ and $d_\mu(x) = t$ for μ-a.e. $x \in Z$, then $\dim_H Z \geq t$.
- If $d_\mu(x) = t$ for **every** $x \in Z$, then $\dim_H Z = t$.

Brin–Katok and Birkhoff: For μ ergodic, $h_\mu(x)$ and $\lambda(x)$ are constant μ-a.e. and equal to $h(\mu)$ and $\lambda(\mu)$.

$$d_\mu(x) = \frac{h(\mu)}{\lambda(\mu)} =: d(\mu) \qquad \mu\text{-a.e.}$$

Not quite enough: we need a measure μ such that $d_\mu(x)$ is constant **everywhere**.

$$d_\mu(x) = \frac{h_\mu(x)}{\lambda(x)} = t \quad \Leftrightarrow \quad h_\mu(x) - t\lambda(x) = 0$$

The Gibbs condition and Bowen's equation

We want a measure μ such that

$$\lim_{n\to\infty} -\frac{1}{n}\log(\mu(B(x,n,\delta)) - t\frac{1}{n}S_n(\log|f'|)(x) = 0$$

for **every** x. Now we are in thermodynamics... Gibbs condition for μ, φ: there exist $M > 0$ and $P = P(\varphi) \in \mathbb{R}$ such that

$$\frac{1}{M} \le \frac{\mu(B(x,n,\delta))}{e^{-nP+S_n\varphi(x)}} \le M$$

for all x, n, δ. For **every** x, this implies

$$\lim_{n\to\infty} -\frac{1}{n}\log(\mu(B(x,n,\delta)) + \frac{1}{n}S_n\varphi(x) = P(\varphi).$$

Thus we need $P(-t\log|f'|) = 0$: Bowen's equation. If this holds and μ is a Gibbs measure for $-t\log|f'|$, then $d_\mu(x) = t$ everywhere, so $\dim_H C = t$.

Conformal repellers: Bowen and Ruelle

General classical setting: M a Riemannian manifold, $V \subset M$ open, $f : V \to M$ conformal and $C^{1+\varepsilon}$. Suppose $J \subset M$ has the following properties:

1. J is compact.
2. J is maximal: $J = \{x \in V \mid f^n(x) \in V \text{ for all } n\}$.
3. f is topologically mixing on J.
4. f is uniformly expanding on J.

Theorem (Ruelle, 1982)

Under these assumptions, there exists a unique $t \in \mathbb{R}$ such that $P(-t \log \|Df\|) = 0$. This t is the Hausdorff dimension of J.

A multifractal decomposition

Given $\alpha \in \mathbb{R}$, consider $K_\alpha^{\mathcal{L}} = \{x \mid \lambda(x) = \alpha\}$.

Let $X' = \{x \mid \lambda(x) \text{ does not exist}\}$. Then

$$J = \left(\bigcup_{\alpha \in \mathbb{R}} K_\alpha^{\mathcal{L}} \right) \cup X'$$

is a *multifractal decomposition* of the repeller J. Every ergodic measure μ is supported on some $K_\alpha^{\mathcal{L}}$.

Observation: Let μ be the Gibbs measure for $-t \log |f'|$, where $t = \dim_H J$, and let $\alpha = \lambda(\mu)$. Then $\mu(K_\alpha^{\mathcal{L}}) = 1$, and so $\dim_H K_\alpha^{\mathcal{L}} = \dim_H J$.

What about the level sets $K_\alpha^{\mathcal{L}}$ for other values of α? What is their Hausdorff dimension?

Beyond Gibbs measures

Goal: Use Bowen's equation to determine Hausdorff dimension of non-compact sets, such as $K_\alpha^{\mathcal{L}}$.
Problem: The classical thermodynamic formalism says nothing about the existence of Gibbs measures on such sets.
Solution: Barreira and Schmeling give a proof that does not involve measures. This removes the following hypotheses from Ruelle's result: compactness, maximality, topological mixing.
The following hypotheses remain: conformality, uniform expansion.

- We must keep conformality or else deal with the fact that $\log \|Df^n(x)\|$ is no longer a Birkhoff sum in the non-conformal case.
- Uniform expansion can be significantly weakened, provided we still have bounded distortion and asymptotic exponential expansion.

Beyond uniform expansion

Given a conformal map $f\colon X \to X$ and a set $Z \subset X$, consider the following hypotheses:

1. f has no critical points ($\|Df(x)\|$ is non-vanishing).
2. Every point $x \in Z$ has *tempered contraction*: for every $\varepsilon > 0$ we have $\inf_{0 \leq k \leq n < \infty}(\log \|Df^{n-k}(f^k(x))\| + n\varepsilon) > -\infty$.
   (Unbounded contraction does not happen too quickly along an orbit: automatic if $\underline{\lambda}(x) = \overline{\lambda}(x)$ or if $\|Df(x)\| \geq 1$ everywhere.)
3. Every point $x \in Z$ has $0 < \underline{\lambda}(x) \leq \overline{\lambda}(x) < \infty$.

Theorem (C., 2009)

Under the above hypotheses,

$$\begin{aligned}\dim_H Z &= \inf\{t \mid P_Z(-t\log\|Df\|) \leq 0\}\\ &= \sup\{t \mid P_Z(-t\log\|Df\|) > 0\}.\end{aligned}$$

Application to multifractal analysis

Let $f\colon X \to X$ be conformal. Two multifractal spectra:

Entropy spectrum for Lyapunov exponents: $\mathcal{L}_E(\alpha) = h_{\text{top}}\, K_\alpha^{\mathcal{L}}$

Dimension spectrum for Lyapunov exponents: $\mathcal{L}_D(\alpha) = \dim_H K_\alpha^{\mathcal{L}}$

One can show that $P_{K_\alpha^{\mathcal{L}}}(-t \log \|Df\|) = h_{\text{top}}\, K_\alpha^{\mathcal{L}} - t\alpha$. For $\alpha > 0$, $K_\alpha^{\mathcal{L}}$ satisfies the second and third hypotheses of the theorem. Thus $\mathcal{L}_D(\alpha) = \frac{1}{\alpha}\mathcal{L}_E(\alpha)$ provided f has no critical points.

How do we get $\mathcal{L}_E(\alpha)$? This takes us deeper into multifractal analysis...

Other multifractal spectra: the Birkhoff spectrum

General scheme: quantify level sets of local quantity ($\lambda(x)$, $h_\mu(x)$, $d_\mu(x)$) using global dimensional quantity ($\dim_H$, h_{top}).

Lyapunov spectrum is particular case of a more general spectrum.

Given $\varphi\colon X \to \mathbb{R}$,

$$K_\alpha^{\mathcal{B}} = \left\{ x \in X \;\middle|\; \lim_{n\to\infty} \frac{1}{n} S_n\varphi(x) = \alpha \right\}.$$

The Birkhoff spectrum of φ is

$$\mathcal{B}(\alpha) = h_{\text{top}}\, K_\alpha^{\mathcal{B}}.$$

Guiding philosophy: Obtain $\mathcal{B}(\alpha)$ as Legendre transform of $T(q) = P(q\varphi)$. Legendre transform acts on functions:

- Maps convex to concave and vice versa.
- Applying two Legendre transforms gives the convex/concave hull.

A general result

Known results: Work with particular class of systems, use tools specific to that class to analyse first $T(q)$ and then $\mathcal{B}(\alpha)$.

Theorem (C., 2009)

Let X be a compact metric space, $f: X \to X$ be continuous, and $\varphi: X \to \mathbb{R}$ be continuous.

1. $T(q) = \sup_{\alpha\in\mathbb{R}}(\mathcal{B}(\alpha) + q\alpha)$.
2. *If $T(q)$ is differentiable and equilibrium states exist for each q, then*
$$\mathcal{B}(\alpha) = \inf_{q\in\mathbb{R}}(T(q) - q\alpha).$$
In particular, $\mathcal{B}(\alpha)$ is concave and differentiable.

T non-differentiable at q: phase transition. We get partial results for the part of the spectrum away from the phase transition.

Examples

Lyapunov spectra:

- Uniformly expanding conformal repeller (Weiss).
- Non-uniformly expanding interval maps: possible phase transitions (Pollicott–Weiss, Nakaishi, Gelfert–Rams).

Birkhoff spectrum: (no requirement of conformality)

- Uniformly hyperbolic map, Hölder continuous potential (Pesin–Weiss).
- Uniformly hyperbolic map, non-Hölder potential: possible phase transitions (thermodynamics by Sarig, Hu, Pesin–Zhang).
- Unimodal interval maps, Hölder continuous potentials with $\sup \varphi - \inf \varphi < h_{\text{top}}(f)$ (thermodynamics by Bruin–Todd).

Phase transitions

How to handle phase transitions? Approximation from within by subsystems with no phase transitions.

Theorem (C., 2009)

Let f, X, φ be as before. Suppose that there exists a sequence of compact f-invariant subsets $X_n \subset X$ such that

1. $q \mapsto P_{X_n}(q\varphi)$ *is differentiable for all* q;
2. *equilibrium states exist for all* q; *and*
3. $\lim_{n\to\infty} P_{X_n}(q\varphi) = P_X(q\varphi)$ *for all* q.

Then $\mathcal{B}(\alpha) = \inf_{q\in\mathbb{R}}(T(q) - q\alpha)$ *for all* α.

This theorem applies to any potential φ on a uniformly hyperbolic map which is Hölder continuous except at some finite number of periodic points.