THERMODYNAMIC FORMALISM

SETTING: X a cpt metric space, $f : X \circlearrowleft$ cts

$$\begin{tikzcd}
& \boxed{\text{topological/smooth dynamics}} \arrow[dl, leftrightarrow, "\text{variational principles}"'] \arrow[dr, leftrightarrow, "\text{multifractals}"] & \\
\boxed{\text{ergodic theory}} \arrow[rr, leftrightarrow, "\text{thermodynamics}"] & & \boxed{\text{dimension theory}}
\end{tikzcd}$$

GOAL:
1. Given a topological / smooth dynamical system, understand the space of invariant measures so we can do ergodic theory.
2. Obtain inv. measures with statistical properties that capture the mixing behaviour of the topological system.
3. Understand the phase space of the system in terms of local asymptotic quantities.

MAJOR THEME: Use tools from statistical mechanics & information theory to relate the 3 areas above.

Outline

I. Fundamentals
 A. Topological entropy, variational principle
 B. Topological pressure, equilibrium states, Gibbs measures

II. The big picture
 A. Pressure as a function, Legendre transform
 B. Multifractals, consequences of existence of a unique equilibrium state

III. Examples & approaches to existence & uniqueness
 A. Uniform hyperbolicity, Markov partitions
 B. Non-uniform hyperbolicity, inducing schemes
 C. Axiomatic approach, specification
 D. Functional analysis, asymmetric Sobolev spaces

IV. Applications, properties of equilibrium states
 A. SRB measures
 B. Statistical properties

I. Fundamentals

A. Topological entropy, variational principle

① Capacity entropy (classical defn). Given $\delta > 0$, $n \in \mathbb{N}$, let $Q(n,\delta) = \min\{\# E \mid \bigcup_{x\in E} B(x,n,\delta) = X\}$,

$$\underline{Ch}_{top}(X,\delta) = \varliminf_{n\to\infty} \frac{1}{n}\log Q(n,\delta),$$
$$\overline{\underline{Ch}}_{top}(X) = \lim_{\delta\to 0} \overline{\underline{Ch}}_{top}(X,\delta).$$

$\boxed{\text{Rmk}}$ Let $R(n,\delta) = \max\{\# F \mid x \neq y \in F \Rightarrow y \notin B(x,n,\delta)\}$; then each such F is (n,δ)-spanning if maximal, so

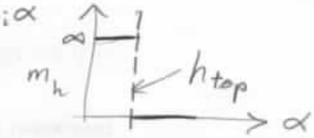
$$R(n,\delta) \geq Q(n,\delta) \geq R(n,\delta/2)$$

Thus R or Q may be used for $\overline{\underline{Ch}}_{top}(X)$.

$\boxed{\text{Rmk}}$ Same procedure with any $Z \subset X$ gives $\overline{\underline{Ch}}_{top}(Z)$.

② Caratheodory entropy (Bowen's defn). Given $Z \subset X$, $\delta > 0$, $N \in \mathbb{N}$, let $\mathcal{P}(Z,\delta,N) = \{\{(x_i,n_i)\}_i \mid x_i \in Z,\ n_i \geq N,\ Z \subset \bigcup_i B(x_i,n_i,\delta)\}$

For every $\alpha \geq 0$, let $m_h(Z,\alpha,\delta,N) = \inf_{\mathcal{P}(Z,\delta,N)} \sum_i e^{-n_i\alpha}$



$$m_h(Z,\alpha,\delta) = \lim_{N\to\infty} m_h(Z,\alpha,\delta,N)$$
$$h_{top}(Z,\delta) = \sup\{\alpha \geq 0 \mid m_h(Z,\alpha,\delta) = \infty\}$$
$$h_{top}(Z) = \lim_{\delta\to 0} h_{top}(Z,\delta)$$

$\boxed{\text{Rmk}}$ The map f is implicit in all this, although not written.

③ General properties. Usually we care about invariant Z, so $Ch_{top}(Z)$ exists. If Z cpt & inv, then $h_{top}(Z) = Ch_{top}(Z)$.

$\boxed{\text{Example}}$ $X = \Sigma_2^+$, $f = \sigma$, $\alpha \in [0,1]$, $Z = \{x \mid$ asymptotic frequency of 0s equal to $\alpha\}$. Then $h_{top}(Z) < Ch_{top}(Z) = \log 2$ $\forall \alpha \neq \frac{1}{2}$.
(Return to this later – multifractals.)

Standard dimensional fact: $h_{top}(\bigcup_n Z_n) = \sup_n h_{top}(Z_n)$

④ Characterising measures. $h(\mu) = \inf\{h_{top}(Z) \mid \mu(Z) = 1\}$.

$$\underline{h}_\mu^\alpha(x) = \lim_{\delta\to 0}\varlimsup_{n\to\infty} \frac{\alpha\log\mu(B(x,n,\delta))}{\log(\xi(x,n)\eta(x,n)^\alpha)} = \lim_{\delta\to 0}\varlimsup_{n\to\infty} -\frac{1}{n}\log\mu(B(x,n,\delta))$$

is independent of α, so write $h_\mu(x)$.

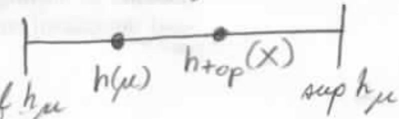
$\mu \in \mathcal{M}_e^f(X) = \{\mu \in \mathcal{M}(X) \mid \mu \text{ ergodic \& } f\text{-inv.}\} \Rightarrow h_\mu(x) = h_\mu(f)$ for μ-a.e. $x \in X$, by Brin–Katok, hence $h(\mu) = h_\mu(f)$.

$\boxed{\text{Rmk}}$ $h(\mu)$ (Caratheodory) and $h_\mu(f)$ (Kolmogorov–Sinai) behave differently for non-ergodic measures: if $\mu = t\nu_1 + (1-t)\nu_2$, then $\mu(Z) = 1 \Leftrightarrow \nu_1(Z) = \nu_2(Z) = 1$, and hence $\{Z \mid \mu(Z) = 1\} = \{Z \mid \nu_1(Z) = 1\} \cap \{Z \mid \nu_2(Z) = 1\}$

$\therefore$ $h(\mu) = \max\{h(\nu_1), h(\nu_2)\}$, while $h_\mu(f) = t h_{\nu_1}(f) + (1-t) h_{\nu_2}(f)$.

5 $h(\mu)$ and $h_{top}(X)$. The defn gives $h(\mu) \le h_{top}(X)$ $\forall \mu \in \mathcal{M}^f(X)$. This relied on local entropy being constant μ-a.e. Given information at <u>every</u> point, get upper bound on $h_{top}(X)$:

$$h_{top}(X) \le \sup_{x \in X} \overline{h}_\mu(x)$$



Pf Let $\beta > \sup_{x\in X} \overline{h}_\mu(x)$, $X_N = \{x \in X \mid \mu(B(x,n,\delta)) \ge e^{-n\beta} \ \forall\, n \ge N\}$. Then $F_n \subset X_N$ (n,δ)-separated $\Rightarrow \# F_n \le e^{n\beta}$, hence

$$h_{top}(X) = \sup_N h_{top}(X_N) \le \sup_N \underline{Ch}_{top}(X_n) \le \beta$$

$\boxed{\text{Rmk}}$ Works for $\underline{h}_\mu$ as well, but harder. Moral of story: narrow range of local entropies $\Rightarrow$ high entropy.

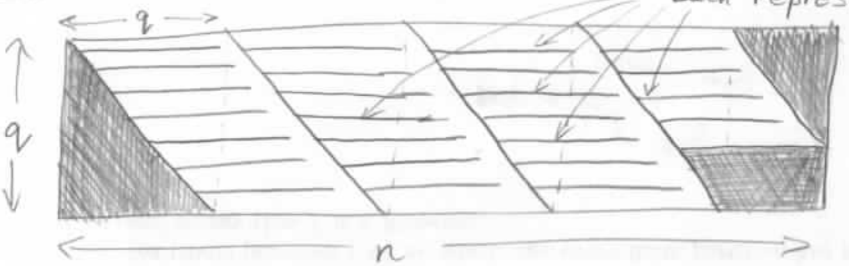
6 Measures of large entropy. How to build a measure with $h(\mu) \approx h_{top}(X)$? Start with atomic measures on pieces of trajectories beginning in an (n,δ)-separated set: fix $\delta > 0$ and $\forall\, n \in \mathbb{N}$, let F_n be a maximal (n,δ)-separated set. Then $\nu_n = \frac{1}{\# F_n} \sum_{x \in F_n} \delta_x$, $\mu_n = \frac{1}{n} \sum_{k=0}^{n-1} \nu_n \circ f^{-k}$, and choose n_j st. $\mu_{n_j} \to \mu \in \mathcal{M}^f(X)$.

$\boxed{\text{Proposition}}$ $h_\mu(f) \ge h_{top}(X, \delta)$ (μ may not be ergodic)

(Pf) Let ξ be a partition of X st. diam $\xi < \delta$, and let $\xi^n = \bigvee_{k=0}^{n-1} f^{-k}\xi$. Then $H_{\nu_n}(\xi^n) = \log R(n,\delta)$.

We want to estimate $H_{\mu_n}(\xi^q)$ for $q < n$ by some combinatorial trickery.

Consider q copies of $\{0,1,2,\dots,n-1\}$, and partition them as shown:



Recall that $H_{\cdot}(\eta_1 \vee \eta_2) \le H_{\cdot}(\eta_1) \vee H_{\cdot}(\eta_2)$, and so

shaded region

$$q \log R(n,\delta) = q H_{\nu_n}(\xi^n) \le \left(\sum_{j=0}^{n-q-1} H_{\nu_n}(f^{-j}\xi^q)\right) + K(q)$$

$$= n\left(\sum \tfrac{1}{n} H_{\nu_n \circ f^{-j}}(\xi^q)\right) + K(q)$$

$$\le n H_{\mu_n}(\xi^q) + K'(q) \quad \text{(by convexity)}$$

This gives $\frac{1}{q} H_{\mu_n}(\xi^q) \ge \frac{1}{n}\log R(n,\delta) - \frac{K'(q)}{qn}$, and taking a limit along $n_j \to \infty$, we get

$$\tfrac{1}{q} H_\mu(\xi^q) \ge h_{top}(X,\delta).$$

Sending $q \to \infty$ finishes the job. ∎

(7) The symbolic case. Let $X = \Sigma_A^+$ be a top. mix. SFT, so Bowen balls $B(x,m,\delta)$ are just cylinders $C_m(x)$. What is $\mu(C_m(x))$? Need to estimate $\mu_n(C_m(x))$. Let $k \in \mathbb{N}$ be such that $f^{k+m}(C_m(x)) = X$ for all $x \in X$, and write $y \vee z = \min\{j \mid y_j \ne z_j\}$.

$\boxed{\text{Lemma}}$ $R(n_1+n_2) \le R(n_1)R(n_2) \le R(n_1+n_2+k)$

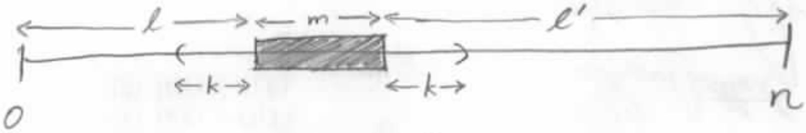
(Pf) Define a map $\{(n_1+n_2)\text{-cyls}\} \to \{n_1\text{-cyls}\} \times \{n_2\text{-cyls}\}$ by $C_{n_1+n_2}(x) \longmapsto (C_{n_1}(x), C_{n_2}(\sigma^{n_1}(x)))$, and another map $\{n_1\text{-cyls}\} \times \{n_2\text{-cyls}\} \to \{(n_1+n_2+k)\text{-cyls}\}$ by $(C_{n_1}(x), C_{n_2}(y)) \longmapsto C_{n_1+n_2+k}(z)$, where $z \in C_{n_1}(x) \cap f^{-(n_1+k)} C_{n_2}(y)$.

Both maps are 1-1, which suffices. ∎

$\boxed{\text{Lemma}}$ Given $k < \ell < n-m-k$, $\ell' = n-m-\ell$, and $x \in X$, we have

$$\frac{R(\ell-k)\,R(\ell'-k)}{R(n)} \leq \nu_n \circ \sigma^{-\ell}(C_m(x)) \leq \frac{R(\ell)\,R(\ell')}{R(n)}$$

(Pf)



Given any $y, y' \in X$, $\exists\, z \in X$ s.t. $z \in C_{\ell-k}(y)$, $\sigma^{\ell}(z) \in C_m(x)$, and $\sigma^{\ell+m+k}(z) \in C_{\ell'}(y')$. This implies that

$$\#\{z \in F_n \mid \sigma^{\ell}(z) \in C_m(x)\} \geq R(\ell-k)\cdot R(\ell'-k),$$

which gives the first inequality. For the second, observe that $y \neq z \in F_n \cap \sigma^{-\ell}(C_m(x)) \Rightarrow y \vee z \leq \ell$ or $\sigma^{\ell+m}(y) \vee \sigma^{\ell+m}(z) \leq \ell'$, hence the map $F_n \to F_\ell \times F_{\ell'}$ given by $y \mapsto (y, \sigma^{\ell+m}(y))$ is 1-1. ∎

Putting the two lemmas together yields

$$\frac{R(n-m-2k)}{R(n)} \leq \nu_n \circ \sigma^{-\ell}(C_m(x)) \leq \frac{R(n-m+k)}{R(n)},$$

and summing over $k < \ell < n-m-k$ gives

$$\frac{R(n-m-2k)}{R(n)}\left(1-\frac{k}{n}\right) \leq \mu_n(C_m(x)) \leq \frac{R(n-m+k)}{R(n)}\left(1-\frac{k}{n}\right) + \frac{k}{n}.$$

Take the limit as $n \to \infty$ (along n_j if need be) to obtain

NEED Lem ‡ FROM P. 9 - RMK ON GROWTH

$$e^{-(m+2k)h_{top}(X)} \leq \mu(C_m(x)) \leq e^{-(m-k)h_{top}(X)}.$$

Another limit as $m \to \infty$ gives $h_\mu(x) = h_{top}(X)$ for <u>every</u> $x \in X$. (Recall comments in ③).

$\boxed{\text{Rmk}}$ For a SFT, the measure we constructed is called the <u>Parry measure</u>, and is usually obtained in a different manner, via the Perron-Frobenius theorem. This approach generalises to a broader setting, which we will consider later.

$\boxed{\text{Rmk}}$ The same proofs work for any map with <u>specification</u>.

B. Topological pressure, equilibrium states, Gibbs measures.

① Capacity pressure (classical def'n). Let $\varphi: X \to \mathbb{R}$ be a continuous potential function. Given $\delta > 0$, $n \in \mathbb{N}$, consider the partition function $Z(n,\delta) = \inf\left\{\sum_{x\in E} e^{S_n\varphi(x)} \,\middle|\, E \text{ } (n,\delta)\text{-spanning}\right\}$,

and let $\overline{CP}_X(\varphi,\delta) = \varlimsup_{n\to\infty} \frac{1}{n}\log Z(n,\delta)$,

$$\overline{CP}_X(\varphi) = \lim_{\delta\to 0} \overline{CP}_X(\varphi,\delta).$$

$\boxed{\text{Rmk}}$ As with $\overline{Ch}_{top}$, can use (n,δ)-separated sets, and can also consider $\overline{CP}_Z(\varphi)$ for $Z \subset X$.

② Carathéodory pressure (Pesin, Pitskel'). Given $Z \subset X$, $\delta > 0$, and $N \in \mathbb{N}$, let $\mathcal{P}(Z,\delta,N)$ be as before, and for any $\alpha \in \mathbb{R}$, let

$$m_P(Z,\varphi,\alpha,\delta,N) = \inf_{\mathcal{P}(Z,\delta,N)} \sum_i e^{-n_i\alpha + S_{n_i}\varphi(x_i)}$$

$$m_P(Z,\varphi,\alpha,\delta) = \lim_{N\to\infty} m_P(Z,\varphi,\alpha,\delta,N)$$

$$P_Z(\varphi,\delta) = \sup\{\alpha\in\mathbb{R} \mid m_P(Z,\varphi,\alpha,\delta) = \infty\}$$

$$P_Z(\varphi) = \lim_{\delta\to 0} P_Z(\varphi,\delta)$$

③ General properties. Reduces to entropy in case $\varphi \equiv 0$, and has similar properties in general: $P_Z = \underline{CP}_Z = \overline{CP}_Z$ if Z is cpt & inv, and P_Z is countably stable.

$\boxed{\text{Proposition}}$ If $a < \varliminf \frac{1}{n} S_n\varphi(x) \le \varlimsup \frac{1}{n} S_n\varphi(x) < b$ $\forall x \in Z$, then $h_{top}(Z) + a \le P_Z(\varphi) < h_{top}(Z) + b$.

(Pf) Given $N \in \mathbb{N}$, let $Z_N = \{x \in Z \mid a < \frac{1}{n} S_n\varphi(x) < b \ \forall n \ge N\}$, and observe that $Z = \bigcup_N Z_N$, so it suffices to prove the result for each Z_N. Now $\forall \delta > 0$, $N' \ge N$, $\alpha \in \mathbb{R}$, we have

$$m_h(Z_N, \alpha - a, \delta, N') \le m_P(Z_N, \alpha, \varphi, \delta, N') \le m_h(Z_N, \alpha - b, \delta, N')$$

and the result follows. ∎

$\boxed{\text{Corollary}}$ If $\frac{1}{n} S_n\varphi(x) \to a$ $\forall x \in Z$, then $P_Z(\varphi) = h_{top}(Z) + a$.

$\boxed{\text{Rmk}}$ Same pf shows that $P_Z(\psi) + a < P_Z(\psi + \varphi) < P_Z(\psi) + b$.

4 Characterising measures. $P_\mu(\varphi) = \inf\{P_Z(\varphi) \mid \mu(Z) = 1\}$.
If μ ergodic, then $\frac{1}{n} S_n\varphi(x) \to \int \varphi\,d\mu$ for all $x \in Z$, $\mu(Z)=1$,
hence $P_Z(\varphi) = h_{top}(Z) + \int\varphi\,d\mu$, so $P_\mu(\varphi) = h(\mu) + \int\varphi\,d\mu$.
Alternately, one may observe that the relevant local quantity is

$$\underline{P}^{\alpha}_{\varphi,\mu}(x) = \lim_{\delta\to 0} \overline{\lim_{n\to\infty}} \frac{\alpha \log \mu(B(x,n,\delta))}{-n\alpha + S_n\varphi(x)}$$

and hence for μ-a.e. x, $\overline{P}^{\alpha}_{\varphi,\mu}(x) = \frac{\alpha h(\mu)}{\alpha - \int\varphi\,d\mu}$.

Setting this equal to α gives $\alpha = h(\mu) + \int\varphi\,d\mu$. (free energy).

5 $P_\mu(\varphi)$ and $P_X(\varphi)$. The def'n gives $h(\mu) + \int\varphi\,d\mu \le P(\varphi)$.
Once again, local information everywhere gives the other bound:

$$P_X(\varphi) \le \sup_{x\in X} \lim_{\delta\to 0} \overline{\lim_{n\to\infty}} \frac{1}{n}\left(-\log\mu(B(x,n,\delta)) + S_n\varphi(x)\right)$$

Pf Given $\beta > \sup$, let $X_N = \{x \in X \mid \mu(B(x,n,\delta)) \ge e^{-n\beta + S_n\varphi(x)}\}$,
so for any (n,δ)-sep set $F_n \subset X_n$, $\sum_{x\in F_n} e^{S_n\varphi(x)} \le e^{n\beta} \sum_x \mu(B(x,n,\delta))$
$\le e^{n\beta}$, $\therefore$ $P_X(\varphi) = \sup_N P_{X_N}(\varphi) \le \sup_N \underline{CP}_{X_N}(\varphi) \le \beta$.

$\boxed{\text{Def'n}}$ $h(\mu) + \int\varphi\,d\mu = P(\varphi) \Rightarrow$ equilibrium state

$\lim_{\delta\to 0} \lim_{n\to\infty} \frac{1}{n}(-\log\mu(B(x,n,\delta)) + S_n\varphi(x)) = P(\varphi)$ everywhere
$\Rightarrow$ (weak) Gibbs measure.

$\boxed{\text{Rmk}}$ Usual def'n of Gibbs measure: $\exists\, C > 0$ s.t.

$$\frac{1}{C} \le \frac{\mu(B(x,n,\delta))}{e^{-nP + S_n\varphi(x)}} \le C \quad \forall\, x \in X,\ n \in \mathbb{N}.$$

This implies uniform convergence with rate $\frac{\log C}{n}$, whereas we only require ptwise convergence with any rate.

6 Measures of large free energy. Modify construction for entropy:
F_n = maximal (n,δ)-separated set, $\nu_n = \frac{1}{Z(n,\delta)} \sum_{x\in F_n} e^{S_n\varphi(x)}\delta_x$.
$\mu_n = \frac{1}{n}\sum_{k=0}^{n-1} \nu_n \circ f^{-k}$, $\mu_{n_j} \to \mu \in \mathcal{M}^f(X)$.

$\boxed{\text{Proposition}}$ $h_\mu(f) + \int \varphi \, d\mu \geq P_X(\varphi, \delta)$

Pf ξ, ξ^n as before, so $H_{\nu_n}(\xi^n) + \int S_n\varphi \, d\nu_n = \log Z(n,\delta)$

(uses fact that $\sum p_i(a_i - \log p_i) = \sum p_i \log \frac{e^{a_i}}{p_i} \leq \log \sum e^{a_i}$ on simplex, with equality iff $p_i = e^{a_i}/\sum e^{a_j}$). Using combinatorics as before,

$$q \log Z(n,\delta) \leq n H_{\mu_n}(\xi^q) + K'(q) + q\int S_n\varphi \, d\nu_n$$

$$\frac{1}{n}\log Z(n,\delta) \leq \frac{1}{q} H_{\mu_n}(\xi^q) + \frac{K'(q)}{qn} + \int \frac{1}{n} S_n \varphi \, d\nu_n$$

$$= \left(\frac{1}{q} H_{\mu_n}(\xi^q) + \int \varphi \, d\mu_n\right) + \frac{K'(q)}{qn}$$

∎

(7) The symbolic case. $X = \Sigma_A^+$ top. mix: $f^{k+m}(C_m(x)) = X$. Want relationship between $Z(n_1+n_2)$ and $Z(n_1)Z(n_2)$ – for entropy, was given by map $F_{n_1+n_2} \longrightarrow F_{n_1} \times F_{n_2}$, but here, <u>value of function matters</u>. We take $x \longmapsto (x, \sigma^{n_1}(x))$, but why should $x \in F_{n_1}$ or $\sigma^{n_1}(x) \in F_{n_2}$ for the choices of F_{n_1} and F_{n_2} that get close to the supremum? Instead, we must take $x \longmapsto (g_{n_1}(x), g_{n_2}(\sigma^{n_1}(x)))$, where $\{g_n(y)\} = C_n(y) \cap F_n$.

Thus, must control $|S_n\varphi(x) - S_n\varphi(y)|$, $x \vee y \geq n$.

$\boxed{\text{Def'n}}$ $V_n(\varphi) = \sup\{|\varphi(x) - \varphi(y)| : x \vee y \geq n\}$

φ is <u>Hölder</u> if $\exists$ $C > 0$, $\theta > 0$ s.t. $V_n(\varphi) \leq C e^{-\theta n}$.

For Hölder φ, get $V_n(S_n\varphi) \leq \sum_{k=0}^{n} C e^{-\theta n} \leq \frac{C}{1-e^{\theta}} =: M$

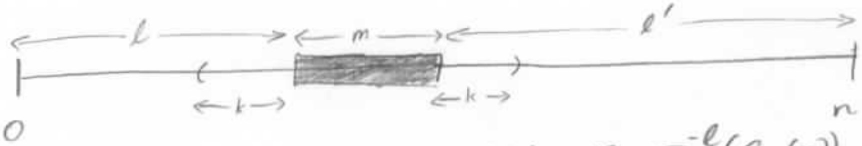
$\boxed{\text{Lemma}}$ φ Hölder $\Rightarrow Z(n_1+n_2)e^{-2M} \leq Z(n_1)Z(n_2) \leq Z(n_1+n_2+k)e^{2M-M'}$

Pf Define $F_{n_1+n_2} \longrightarrow F_{n_1} \times F_{n_2}$ as above, observe that

$$\sum_{x \in F_{n_1+n_2}} e^{S_{n_1+n_2}\varphi(x)} = \sum_{x \in F_{n_1+n_2}} e^{S_{n_1}\varphi(x)} e^{S_{n_2}\varphi(\sigma^{n_1}(x))}$$

$$\leq \left(\sum_{y \in F_{n_1}} e^{S_{n_1}\varphi(y)} e^{M}\right)\left(\sum_{z \in F_{n_2}} e^{S_{n_2}\varphi(z)} e^{M}\right)$$

Similarly for $F_{n_1} \times F_{n_2} \longrightarrow F_{n_1+n_2+k}$, with $M' = k \inf \varphi$. ∎



$\boxed{\text{Lemma}}$ $\dfrac{Z(\ell-k)Z(\ell'-k)}{Z(n)}e^{-3M+2M'} \le \dfrac{\nu_n\circ\sigma^{-\ell}(C_m(x))}{e^{S_m\varphi(x)}} \le \dfrac{Z(\ell)Z(\ell')}{Z(n)}e^{3M}$

(Pf) Use onto map $F_n \cap \sigma^{-\ell}(C_m(x)) \to F_{\ell-k}\times F_{\ell'-k}$ to get

$$Z(n)\nu_n\circ\sigma^{-\ell}(C_m(x)) = \sum_{\substack{y\in F_n\\ \sigma^\ell y\in C_m(x)}} e^{S_n\varphi(y)} \ge \sum e^{S_{\ell-k}\varphi(y)}e^{S_m\varphi(\sigma^\ell y)}e^{S_{\ell'-k}\varphi(\sigma^{\ell+m+k}y)}e^{2M'}$$

$$\ge Z(\ell-k)e^{S_m\varphi(x)}Z(\ell'-k)e^{-3M+2M'}$$

Similarly, use 1-1 map $F_n\cap\sigma^{-\ell}(C_m(x)) \to F_\ell\times F_{\ell'}$ to get

$$Z(n)\nu_n\circ\sigma^{-\ell}(C_m(x)) \le \sum_{\substack{y\in F_n\\ \sigma^\ell y\in C_m(x)}} e^{S_\ell\varphi(y)}e^{S_m\varphi(\sigma^\ell(y))}e^{S_{\ell'}\varphi(\sigma^{\ell+m}(y))}$$

$$\le Z(\ell)e^{S_m\varphi(x)}Z(\ell')e^{3M} \qquad ■$$

Combining the lemmas and summing over ℓ gives

$$(*)\quad \frac{Z(n-m-2k)}{Z(n)}e^{-5M+2M'}\left(1-\frac{k}{n}\right) \le \frac{\mu_n(C_m(x))}{e^{S_m\varphi(x)}} \le \frac{Z(n-m+k)}{Z(n)}e^{5M-M'}\left(1-\frac{k}{n}\right)+\frac{k}{n}$$

(‡) $\boxed{\text{Lemma}}$ Let $\{a_n\}$ be such that $|a_{m+n}-a_m-a_n|\le R$, R constant. Then $P=\lim_{n\to\infty}\frac{a_n}{n}$ exists, and $|a_n-Pn|\le R$ $\forall\, n$.

(Pf) Fix m, and let $n=km+\ell$, $0\le\ell<m$. By induction, $|a_{km}-ka_m|\le(k-1)R$, so $|a_n-ka_m-a_\ell|\le kR$, and $\left|\frac{a_n}{n}-\frac{ka_m}{km+\ell}-\frac{a_\ell}{km+\ell}\right|\le\frac{kR}{km+\ell}$.

Let $n\to\infty$, get $\frac{a_m}{m}-\frac{R}{m}\le\overline{\lim_{n\to\infty}}\frac{a_n}{n}\le\frac{a_m}{m}+\frac{R}{m}$. Thus limit exists.

If $a_m\ge mP+\gamma R$ for some $m\in\mathbb{N}$, $\gamma>1$, then $a_{km}\ge ka_m-(k-1)R$ $\ge kmP+[(\gamma-1)k-1]R$ $\forall\, k$, hence $\overline{\lim}\frac{a_n}{n}\ge P+(\gamma-1)R$. ■

From Lemma on p. 8, $\log Z(n_1+n_2)-2M\le\log Z(n_1)+\log Z(n_2)$ $\le\log Z(n_1+n_2+k)+2M-M'\le\log Z(n_1+n_2)+\log Z(k)+4M-M'$

$\therefore$ $\log Z(n)$ satisfies Lemma ‡ with $R=\max(2M,\ 4M-M'+\log Z(k))$.

Consequently, $|\log Z(n)-Pn|\le R$, so $e^{-R}\le\frac{Z(n)}{e^{Pn}}\le e^{R}$ for all n.

$$\frac{Z(n-m-2k)}{Z(n)} \geq e^{P(m+2k)} e^{-2R}$$

$$\frac{Z(n-m+k)}{Z(n)} \leq e^{P(m-k)} e^{2R}$$

$$\Rightarrow e^{2kP - 2R - 5M + 2M'} \leq \frac{\mu(C_m(x))}{e^{-mP + S_m\varphi(x)}} \leq e^{-kP + 2R + 5M - M'}$$

So μ is a Gibbs measure. In particular, local quantity constant everywhere, and local entropy depends only on Birkhoff average of φ.

$\boxed{\text{RMK}}$ Again, all this works for specification as well.

(8) A glance ahead. Let $\varphi_1, \varphi_2 : X \to \mathbb{R}$ be cts. Then $\forall \mu \in \mathcal{M}(X)$, we have $\left|\int \varphi_1 \, d\mu - \int \varphi_2 \, d\mu\right| \leq \|\varphi_1 - \varphi_2\|_{C^0}$, hence if μ is an eq. st. for φ_1, it is nearly an eq. st. for φ_2.

Similarly, if $\varphi_1, \varphi_2 : \Sigma_A^+ \to \mathbb{R}$ are Hölder, and μ is Gibbs for φ_1, then μ is nearly Gibbs for φ_2. Indeed, write $\psi = \varphi_1 - \varphi_2$, then

$$\frac{1}{K} e^{S_m\psi(x)} \leq \frac{\mu(C_m(x))}{e^{-mP + S_m\varphi_2(x)}} \leq e^{S_m\psi(x)} K$$

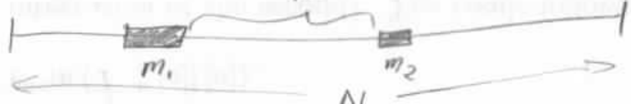
In particular, $P_{\varphi_2,\mu}(x)$ is determined by $\frac{1}{m} S_m \psi(x)$ as $m \to \infty$. We will say more about this quantity later. For now, the moral is that (1) P behaves well as φ varies, (2) varying φ can tell us things about $\mathcal{M}_e^f(X)$.

Addendum to I. A. 7. & I. B. 7.

The Gibbs measures constructed here are mixing and unique.

Mixing (sort of) $\mu\left(C_{m_1}(x) \cap \sigma^{-n} C_{m_2}(x)\right) \to \geq \gamma\, \mu\left(C_{m_1}(x)\right) \mu\left(C_{m_2}(x)\right)$

Pf (Sketch) Either look at



or use Gibbs property and sum over all connecting strings. ∎

Unique If not unique, then $\exists\ \nu \perp \mu$, also eq. st. Let B be st. $\sigma(B) = B$, $\nu(B) = 1$, $\mu(B) = 0$. Any partition has F_n that approximates B eventually.

$$P(\varphi) \approx \frac{1}{n} H_\nu(\xi^n) + \int S_n \varphi \, d\nu \approx \frac{1}{n} \sum_{C \in \xi^n} \nu(C)\left[S_n \varphi(x_C) - \log \nu(C)\right]$$

$$n P(\varphi) \approx \nu(F_n) \sum_{\substack{C \in \xi^n \\ C \subset F_n}} \frac{\nu(C)}{\nu(F_n)}\left[S_n \varphi(x_C) - \log\left(\frac{\nu(C)}{\nu(F_n)}\right) - \log \nu(F_n)\right] + \text{(complement)}$$

Recall: $\sum p_i = 1$, $p_i \geq 0 \Rightarrow$

$$\sum p_i (a_i - \log p_i) = \sum p_i \log\left(\frac{e^{a_i}}{p_i}\right) \leq \log \sum e^{a_i}$$

$$\therefore\ n P(\varphi) \leq \nu(F_n) \log\left(\sum_{\substack{C \in \xi^n \\ C \subset F_n}} e^{S_n \varphi(x_C)}\right) \approx \nu(F_n) \log\left(\sum \mu(C) e^{nP}\right)$$

$$= \nu(F_n) \log\left(\mu(F_n) e^{nP}\right) = nP\,\nu(F_n) + \underbrace{\nu(F_n)}_{\to 1} \underbrace{\log \mu(F_n)}_{\to -\infty}.$$

II. The Big Picture

A. Pressure as a function, Legendre transform, invariant measures

① Legendre transform. V a top. vec sp, $T: V \to \mathbb{R}$ a convex function:

$$T(sv + (1-s)w) \le sT(v) + (1-s)T(w) \quad \forall\, v, w \in V,\ s \in [0,1].$$

$S: V^* \to \mathbb{R}$ is the Leg tr of T (write $S = T^{L_1}$) if

$$S(v^*) = \inf_{v \in V} \left(T(v) - v^*(v)\right) \qquad (*)$$

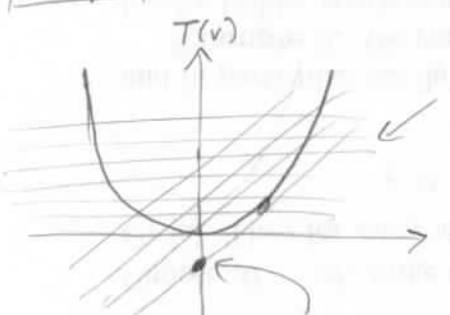
Geometric interpretation: v^* represents codim-1 subspace of V ($\ker v^*$) + scaling factor. How to "see" $T(v) - v^*(v)$?

$\{(w, T(v) - v^*(v-w)) \mid w \in V\}$ = affine subspace of $V \times \mathbb{R}$ parallel to graph v^* through $(v, T(v))$

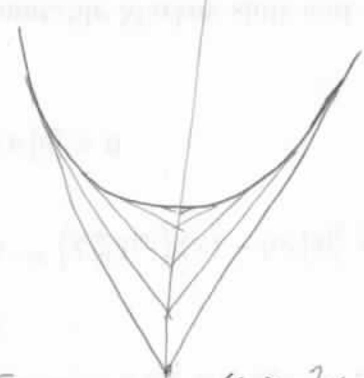
$T(v) - v^*(v)$ = height at which this subspace intersects $\{0\} \times \mathbb{R}$.

$S(v^*) = T^{L_1}(v^*)$ = smallest such value

$\boxed{\text{Example}}$ $V = \mathbb{R} = V^*$ $\quad T(v) = v^2$ $\qquad$ argument of S = <u>slope</u> of T



$v^* = 0,\ S(0) = 0$

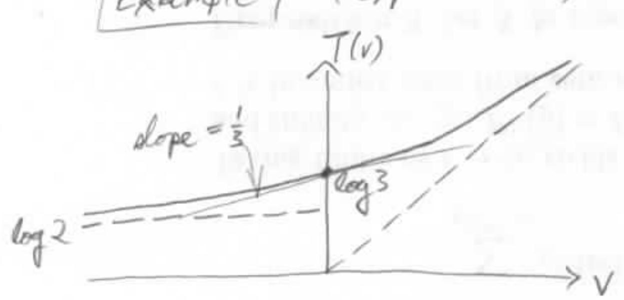


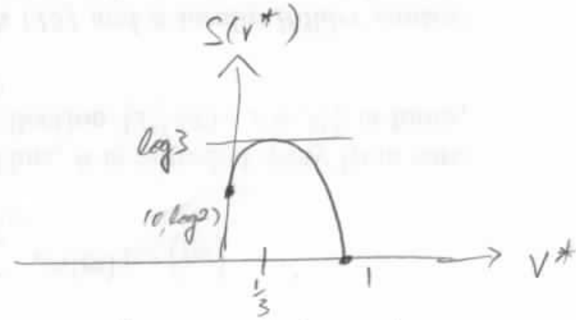
$v^* = 1,\ S(1) = -1$

$C^1 \Rightarrow$ inf happens when $T'(v) = v^*$: in this case $T'(v) = 2v$,

so $S(v^*) = T\left(\frac{v^*}{2}\right) - v^*\left(\frac{v^*}{2}\right) = -\frac{(v^*)^2}{4}$.

$\boxed{\text{Example}}$ (exponential) $T(v) = \log(e^v + 2)$





$S(v^*) = -\infty \quad \forall\ v^* \notin [0,1]$ $\qquad$ slope at endpts is $\pm\infty$

Observe that $T(v) - S(v^*) \geq v^*(v) \quad \forall \ (v, v^*) \in V \times V^*$, with equality iff (v, v^*) achieve the infimum in $(*)$ – in this case (v, v^*) is a Legendre transform pair. If every v is part of such a pair, then $T(V) = \sup_{v^* \in V^*} \left(S(v^*) + v^*(v) \right) = S^{L_2}(v)$

This has a similar geometric interpretation.

$\boxed{Q}$ How to make every v, v^* part of a Leg transform pair?

$\boxed{A}$ (sort of) semi-continuity.

Remark: T, S can be arbitrary, and defns still work. Get

① T^{L_1} is concave / S^{L_2} convex

② $(T^{L_1})^{L_2} = \tilde{T}$ = convex & LSC hull

③ $T^{L_1} = \tilde{T}^{L_1}$ $\quad \left\{ \begin{matrix} \text{LSC} \\ \text{convex} \\ V \to \mathbb{R} \end{matrix} \right\} \underset{L_2}{\overset{L_1}{\rightleftarrows}} \left\{ \begin{matrix} \text{USC} \\ \text{concave} \\ V^* \to \mathbb{R} \end{matrix} \right\}$

② The key example. $V = C(X)$, $P : C(X) \to \mathbb{R}$ pressure

$V^* = C(X)^* \supset M^f(X)$. Variational principle:

$$P(\varphi) = \sup_{\mu \in M^f(X)} \left(h_\mu(f) + \int \varphi \, d\mu \right) = \left(h_\cdot(f) \right)^{L_2}(\varphi)$$

Legendre pairs are equilibrium states. (sort of)

$\boxed{\text{Question}}$ $\quad h_\mu(f) = \inf_{\varphi \in C(X)} \left(P(\varphi) - \int \varphi \, d\mu \right)$ (?) (concave, even affine)

Every μ part of a pair $\iff$ every μ is an eq state for some φ.
In fact, the question (?) doesn't require quite this much.

$\boxed{\text{Theorem}}$ Given $S: V^* \to \mathbb{R}$, $(S^{L_2})^{L_1}$ is the concave <u>and upper semi-continuous</u> hull of S: that is,

$$(S^{L_2})^{L_1}(v^*) = \inf\{R(v^*) \mid R: V^* \to \mathbb{R} \text{ concave \& usc}, R \geq S\}$$
$$= \inf\{R(v^*) \mid R: V^* \to \mathbb{R} \text{ concave \& cts}, R \geq S\}$$

uses Hahn-Banach

$$= \inf\{R(v^*) \mid R: V^* \to \mathbb{R} \text{ affine \& cts}, R \geq S\}$$

$\boxed{\text{Pf}}$ Observe that if $\{R_\alpha\}$ are all concave & usc, then so is $R_0 = \inf_\alpha R_\alpha$. Indeed,

$$R_0(tv^* + (1-t)w^*) = \inf_\alpha R_\alpha(tv^* + (1-t)w^*)$$
$$\geq \inf_\alpha\left(t R_\alpha(v^*) + (1-t) R_\alpha(w^*)\right)$$
$$\geq \inf_\alpha (t R_\alpha(v^*)) + \inf_\alpha((1-t) R_\alpha(w^*))$$
$$= t R_0(v^*) + (1-t) R_0(w^*)$$

and if $v_n^* \to w^*$, then

$$R_0(w^*) = \inf_\alpha R_\alpha(w^*) \geq \inf_\alpha \overline{\lim_{n\to\infty}} R_\alpha(v_n^*)$$
$$\geq \overline{\lim_{n\to\infty}} \inf_\alpha R_\alpha(v_n^*) = \overline{\lim_{n\to\infty}} R_0(v_n^*)$$

Follows from result ~~that~~ if S usc & concave~~, then~~ $S^{L_2L_1} = S$.

Given $v^* \in V^*$, fix $b > S(v^*)$, let $C = \{(w^*, t) \in V^* \times \mathbb{R} \mid 0 \leq t \leq S(w^*)\}$

Hahn-Banach $\Rightarrow \exists$ cts lin fnl $F: V^* \times \mathbb{R} \to \mathbb{R}$ s.t.

$$F(w^*, t) < F(v^*, b) \quad \forall (w^*, t) \in \overline{C} \qquad (\text{NEEDS usc})$$

$F(w^*, t) = w^*(v) + dt$ for some fixed $v \in V$, $d \in \mathbb{R}$. Let $\tilde{v} = \frac{v}{d}$.

$$(v^*(v) + dt < v^*(v) + db \Rightarrow d > 0)$$

$$w^*(\tilde{v}) + t < v^*(\tilde{v}) + b$$

Take supremum over $\overline{C}$:

$$S^{L_2}(\tilde{v}) \leq v^*(\tilde{v}) + b \Rightarrow v^*(\tilde{v}) \geq S^{L_2}(\tilde{v}) - b$$

$$\Rightarrow b \geq (S^{L_2})^{L_1}(v^*)$$

∎

So for upper semi-cts entropy, pressure determines entropy.

~~Furthermore~~, pressure determines $\mathcal{M}^f(X)$: (always)

$\boxed{\text{Thm}}$ X cpt met sp, $f: X \circlearrowleft$ cts, $h_{top} < \infty$. Given $v^* \in C(X)^*$, we have $v^* \in \mathcal{M}^f(X) \iff v^*(\varphi) \le P(\varphi) \quad \forall\ \varphi \in C(X)$

$$\left(\iff \rho^{L_1}(v^*) \ge 0 \right).$$

$\boxed{\text{Pf}}$ ① $\underline{v^*(\varphi) \ge 0}$: Fix $\varepsilon > 0$, $\varphi \ge 0$. Then $\forall\ n \in \mathbb{N}$, we have

$$v^*(n(\varphi+\varepsilon)) = -v^*(-n(\varphi+\varepsilon)) \ge -P_X(-n(\varphi+\varepsilon))$$

$$\ge -\left[h_{top}(X) + \sup_x(-n(\varphi(x)+\varepsilon))\right]$$

$$\ge -h_{top}(X) + n\varepsilon$$

n large $\Rightarrow v^*(n(\varphi+\varepsilon)) \ge 0 \quad \therefore\ v^*(\varphi+\varepsilon) \ge 0$

ε arbitrary $\Rightarrow v^*(\varphi) \ge 0$.

② $\underline{v^*(1) = 1}$: $\quad n v^*(1) \le h_{top}(X) + n \quad \forall\ n \in \mathbb{Z}$

③ $\underline{v^*(\varphi\circ f - \varphi) = 0}$: $\quad n v^*(\varphi\circ f - \varphi) \le P_X(n(\varphi\circ f - \varphi)) = h_{top}(X)$ ■

since pressure of a cobdry is ~~top~~ ent.

③ Differentiability. $v^* \in V^*$ is a $\underline{\text{tangent functional}}$ at $v \in V$ if $T(v+w) \ge T(v) + v^*(w) \quad \forall\ w \in V$.

tangent functionals $\iff$ Legendre transform pairs

If v^* is a tangent functional at v, then

$$\forall\ w \in V, \quad D_w^+ T(v) = \lim_{t\to 0^+} \frac{T(v+tw) - T(v)}{t} \ge \lim_{t \to 0^+} \frac{v^*(tw)}{t} = v^*(w)$$

In fact, $D_w^+ T(v) = \sup\{ v^*(w) \mid v^* \text{ a tangent fnl at } v\}$

(use Hahn–Banach: $D_\cdot^+ T(v)$ is sublinear, so extend v^* from 1-d to all)

In particular, if v^* is the $\underline{\text{unique}}$ ~~tangent~~ fnl at v, then

$D_w^+ T(v) = v^*(w) \quad \forall\ w \in V$, so T is Gateaux diff. (also $\Leftarrow$)

$\boxed{\text{Example}}$ h USC $\iff$ tangent functionals = eq states, so if h USC then unique eq. state $\iff$ Gâteaux diff. pressure and $\frac{d}{dt} P(\varphi + t\psi)\Big|_{t=0} = \int \psi\, d\mu \quad \forall\ \psi \in C(X)$.

$\boxed{\text{Ruelle's formula}}$

Gateaux differentiable: all directional derivatives exist

Fréchet differentiable: linear operator.

Suppose $\forall\ \psi \in C(X)$ small $\exists\ \mu_\psi$ eq st. for $\varphi + \psi$

and also $\lim_{\|\psi\| \to 0} d_{TV}(\mu_\psi, \mu_0) = 0$. Then by properties of

tangent functionals $P(\varphi) + \int \psi \, d\mu_0 \leq P(\varphi + \psi)$

$$P(\varphi + \psi) - \int \psi \, d\mu_\psi \leq P(\varphi)$$

$$\Rightarrow \quad 0 \leq P(\varphi+\psi) - P(\varphi) - \int \psi \, d\mu_0 \leq \int \psi \, d\mu_\varphi - \int \psi \, d\mu_0$$

$$\therefore \quad \frac{|P(\varphi+\psi) - P(\varphi) - \int \psi \, d\mu_0|}{\|\psi\|_{C(X)}} \leq \frac{|\int \psi \, d\mu_\psi \ \int \psi \, d\mu_0|}{\|\psi\|_{C(X)}} \leq d_{TV}(\mu_\psi, \mu_0)$$

$\therefore$ P is Fréchet diff. at φ.

(4) Cross-sections. Can one-dim "slices" of the pressure function tell us anything?

$C(X)$ is a TVS (even a Banach space), so consider 1-d subspace:

$T(q) = P_X(q\varphi)$, $\varphi \in C(X)$ fixed.

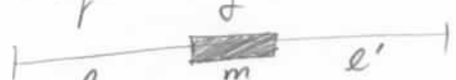
Suppose entropy is USC. Let μ_q = eq state for $q\varphi$. (Not necessarily unique). Then μ_q moves around the simplex $M^f(X)$ through ∞ dimensions - compare with 1-d case.

$\boxed{\text{Example}}$ $X = \Sigma_2^+$, $\varphi = a_0 \mathbb{1}_{[0]} + a_1 \mathbb{1}_{[1]}$

$$Z_n(q\varphi) = \sum_{x_1 \cdots x_n} e^{q(a_{x_1} + \cdots + a_{x_n})}$$

$$= \left(e^{q a_0} + e^{q a_1}\right)^n \Rightarrow P(q\varphi) = \log\left[e^{q a_0} + e^{q a_1}\right]$$

$$\nu_n = \Big(\sum_{x_1 \cdots x_n} e^{q(S_n \varphi(x))} \delta_x\Big) / Z_n$$

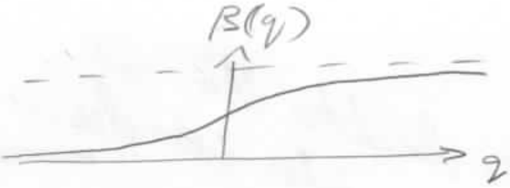


$$\text{Given } 0 \leq m \leq n - \ell, \quad \nu_n(\sigma^{-\ell}(C_m(x))) = \frac{\Big(\sum_{y_1 \cdots y_\ell} e^{q(S_\ell \varphi(y))}\Big) e^{q S_m \varphi(x)} \Big(\sum_{z_1 \cdots z_{\ell'}} e^{q(S_{\ell'} \varphi(z))}\Big)}{Z_n(q\varphi)}$$

$$= \frac{e^{q a_{x_0}} e^{q a_{x_1}} \cdots e^{q a_{x_{m+1}}}}{(e^{q a_0} + e^{q a_1})^m}$$

$$\therefore \mu_q \text{ is } \left(\frac{e^{q a_0}}{e^{q a_0} + e^{q a_1}}, \frac{e^{q a_1}}{e^{q a_0} + e^{q a_1}}\right) \text{ - Bernoulli}$$

Let $\beta(q) = \dfrac{e^{qa_0}}{e^{qa_0}+e^{qa_1}}$ $(a_0 > a_1)$



Then μ_q is $(\beta(q), 1-\beta(q))$ – Bernoulli, and

$$\int \varphi \, d\mu_q = a_0 \beta(q) + a_1 (1-\beta(q)) = T'(q)$$

(cf. Ruelle's formula).

$$K_\alpha = \{ x \in X \mid \tfrac{1}{n} S_n \varphi(x) \longrightarrow \alpha \}$$

$$\alpha(q) = a_0 \beta(q) + a_1 (1-\beta(q)) = T'(q) \Rightarrow \boxed{\mu_q(K_{\alpha(q)}) = 1}$$

$\boxed{\text{Fact}}$ $h(\mu_q) = h_{top}(K_{\alpha(q)})$

(5) Multifractal analysis of Birkhoff averages.

Fix $\varphi : X \to \mathbb{R}$, consider $\mathbb{B}(\alpha) = h_{top}(K_\alpha)$.

Let $T(q) = P_X(q\varphi)$, so $T^{L_1}(\alpha) = \inf_{q \in \mathbb{R}} (T(q) - q\alpha)$

$\boxed{\text{Thm}}$ I. $\mathbb{B} \le T^{L_1}$

II. μ_q an ergodic eq st. for $q\varphi$, $\alpha(q) = \int \varphi \, d\mu_q$

$$\Rightarrow \mathbb{B}(\alpha(q)) = T(q) - q\alpha = T^{L_1}(\alpha(q))$$

(Pf) I. Suffices to show $T(q) = \mathbb{B}^{L_2}(q) = \sup_{\alpha \in \mathbb{R}} (\mathbb{B}(\alpha) + q\alpha)$.

$\le$ is immediate from variational principle.

$\ge$: $F_\alpha^{\varepsilon, N} = \{ x \mid |\tfrac{1}{n} S_n \varphi(x) - \alpha| < \varepsilon \quad \forall \; n \ge N \}$

Fix $\varepsilon > 0$, then $K_\alpha \subset \bigcup_N F_\alpha^{\varepsilon, N}$, so

$$\mathbb{B}(\alpha) = h_{top} K_\alpha = \sup_N h_{top} F_\alpha^{\varepsilon N} \le \sup_N \underline{Ch}_{top} F_\alpha^{\varepsilon, N}$$

As in proof of variational principle, build $\mu_\alpha^{\varepsilon, N}$

s.t. ① $h(\mu_\alpha^{\varepsilon, N}) \ge \underline{Ch}_{top} F_\alpha^{\varepsilon, N}$

② $\left| \int \varphi \, d\mu_\alpha^{\varepsilon, N} - \alpha \right| \le \varepsilon$

$$\therefore T(q) \ge h(\mu_\alpha^{\varepsilon, N}) + \int q\varphi \, d\mu_\alpha^{\varepsilon, N} \ge \underline{Ch}_{top} F_\alpha^{\varepsilon, N} + q\alpha - q\varepsilon.$$

II. $\mathbb{B}(\alpha(q)) = h_{top} K_{\alpha(q)} \ge h(\mu_q) = T(q) - q\alpha(q)$ ∎

In particular, T differentiable $\Rightarrow \alpha(q) = T'(q)$ by Ruelle's formula. The latter takes all values in an interval if T is C^1.

$\boxed{\text{Example}}$ $X = \Sigma_2^+$, $\varphi(x) = \varphi(x_0 x_1) = \varphi_{x_0 x_1}$ $\quad \Phi = \begin{bmatrix} \varphi_{00} & \varphi_{01} \\ \varphi_{10} & \varphi_{11} \end{bmatrix}$

$$Z_n(\varphi) = \sum_{x_0 \cdots x_{n-1}} e^{S_n \varphi(x)} = \sum_{x_0, \dots x_{n-1}} e^{\varphi_{x_0 x_1}} e^{\varphi_{x_1 x_2}} \cdots e^{\varphi_{x_{n-2} x_{n-1}}} e^{\varphi_{x_{n-1} x_n}}$$

$$\approx \sum_{i,j=0}^{1} \left(\begin{bmatrix} e^{\varphi_{00}} & e^{\varphi_{01}} \\ e^{\varphi_{10}} & e^{\varphi_{11}} \end{bmatrix}^n \right)_{ij} \qquad \boxed{\lambda = \text{largest eigval}}$$

$$S \begin{bmatrix} e^{\varphi_{00}} & e^{\varphi_{01}} \\ e^{\varphi_{10}} & e^{\varphi_{11}} \end{bmatrix} S^{-1} = \begin{bmatrix} \lambda & * \\ 0 & * \end{bmatrix}$$

$$= \sum \left(S^{-1} \begin{bmatrix} \lambda^n & * \\ 0 & * \end{bmatrix} S \right)_{ij}$$

(note: can restrict to Σ_A^+ by putting $\varphi_{ij} = -\infty$)

$$\Rightarrow \quad P_X(\varphi) = \log \lambda$$

$$\lambda = \tfrac{1}{2}\left(\mathrm{Tr} + \sqrt{(\mathrm{Tr})^2 - 4\det} \right) = \tfrac{1}{2}\left(e^{\varphi_{00}} + e^{\varphi_{11}} + \sqrt{(e^{\varphi_{00}} - e^{\varphi_{11}})^2 + 4e^{\varphi_{01}+\varphi_{10}}} \right)$$

$$P_X(q\varphi) = -\log 2 + \log\left[e^{q\varphi_{00}} + e^{q\varphi_{11}} + \sqrt{(e^{q\varphi_{00}} - e^{q\varphi_{11}})^2 + 4e^{q(\varphi_{01}+\varphi_{10})}} \right]$$

Note: * $\varphi_{00} = \varphi_{01} = a_0$, $\varphi_{10} = \varphi_{11} = a_1 \Rightarrow P_X(q\varphi) = \log(e^{qa_0} + e^{qa_1})$

* Differentiable, so we get $B(\alpha)$ as T^{L}. (μ_q = eq state)
* What are equilibrium states? As $q \to \infty$, growth may be governed by ① $\varphi_{00} \Rightarrow T(q) \approx q\varphi_{00}$, $\mu_q \to \delta_{\bar{0}}$

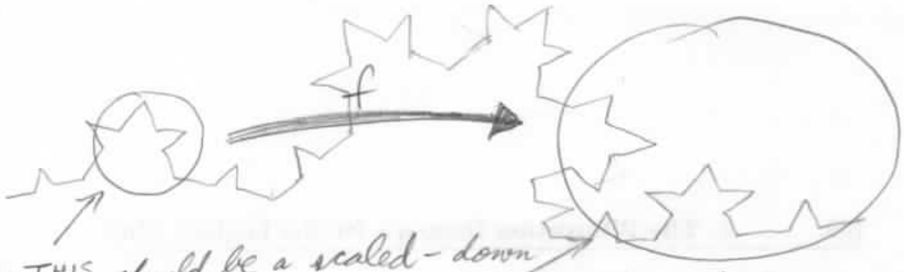
② $\varphi_{11} \Rightarrow T(q) \approx q\varphi_{11}$, $\mu_q \to \delta_{\bar{1}}$

(OR) ③ $\frac{1}{2}(\varphi_{01} + \varphi_{10}) \Rightarrow T(q) \approx \frac{1}{2} q(\varphi_{01} + \varphi_{10})$, $\mu_q \to \frac{1}{2}(\delta_{\overline{01}} + \delta_{\overline{10}})$

So we need Markov measures. But which ones?

(6) Conformal measures. We could build equilibrium measure as limit of δ-measures on (n, ε)-separated orbits (and then later show Gibbs), but there is another way (which gives Gibbs as a natural consequence).

KEY IDEA: Measure should have scaling/self-similarity properties consistent with f and with φ, so build a map on $\mathcal{M}(X)$ for which such a measure is a fixed point.



For some measure, THIS should be a scaled-down copy of THIS, with scaling given some factor ψ.

Defn $\hat{M}(X)$ = all Borel measures on X, $\psi : X \to [0,\infty)$, define $P_\psi : \hat{M}(X) \circlearrowleft$ by
$$(P_\psi \mu)(E) = \int_{f(E)} \psi(f^{-1}(x))\,dx \quad \text{whenever } f \text{ is 1-1 on } E.$$
μ is ψ-conformal if $P_\psi \mu = \mu$.

Rmk $X_\mu = \operatorname{supp}\mu$ is cpt & f-inv, $\forall$ conformal μ (even though μ may not be invariant).

Theorem X a cpt metric space, $f : X \circlearrowleft$ cts, $\psi : X \to (0,\infty)$ cts. Suppose f is non-uniformly expanding: $d(f(x), f(y)) \geq d(x,y)$ $\forall$ suff. close $x, y \in X$. Then μ ψ-conformal $\Rightarrow \mu$ is weak Gibbs for $\log\psi$ on X_μ.

Pf Lemma: $\forall\, \delta > 0\ \exists\, \gamma > 0$ s.t. $\mu(B(x,\delta)) \geq \gamma$ $\forall\, x \in X_\mu$.
Given $\delta > 0$, let $\varepsilon = \varepsilon(\delta) > 0$ be st. $|\log\psi(x) - \log\psi(y)| < \varepsilon$ $\forall\, d(x,y) < \delta$. Now we estimate $\mu(B(x,n,\delta))$:
$$\begin{aligned}\mu(B(x,n,\delta)) &= P_\psi\mu(B(x,n,\delta)) = \int_{\mu(B(f(x),n-1,\delta))} \psi(y)\,d\mu(y) \\ &= e^{\pm\varepsilon}\psi(x)\,\mu(B(f(x),n-1,\delta)) \\ &= \cdots = e^{\pm n\varepsilon}\psi(x)\cdots\psi(f^{n-1}(x))\,\mu(B(f^n(x),\delta))\end{aligned}$$
$$\therefore \left| -\tfrac{1}{n}\log\mu(B(x,n,\delta)) + \tfrac{1}{n}S_n\log\psi(x)\right| < \varepsilon + \frac{|\log\gamma|}{n}$$
Send $n \to \infty$ and $\delta \to 0$ for the result. ▨

Rmk $\mu \to P_\psi\mu / \|P_\psi\mu\|$ is a cts map from $M(X)$ to itself $\therefore$ it has a fixed pt ν. Let $\lambda = \|P_\psi \nu\|$ $\therefore P_\psi\nu = \lambda\nu$, or equivalently, $P_{\psi/\lambda}\nu = \nu$. This seems to say that $P_{X_\nu}(\log\psi) = \log\lambda$ and ν is an equilibrium state — but it may not be invariant!

7 Ruelle-Perron-Frobenius operator. How do we know P_ψ acts continuously on $\hat{M}(X)$? Given $g \in C(X)$, we have

$$\int g \, d(P_\psi \mu) = \lim_{n\to\infty} \sum_k g(x_k^n) P_\psi \mu(E_k^n) \qquad (\xi^n = \{E_k^n\} \text{ a partition})$$

$$= \lim_{n\to\infty} \sum_k g(x_k^n) \int_{f(E_k^n)} \psi(f^{-1}(x)) \, d\mu(x)$$

$$= \lim_{n\to\infty} \sum_k g(x_k^n) \psi(x_k^n) \mu(f(E_k^n))$$

$$= \int \underbrace{\Big(\sum_{fy=x} \psi(y) g(y)\Big)}_{=(\mathcal{L}_\psi g)(x)} d\mu(x)$$

Then $\mathcal{L}_\psi : C(X) \circlearrowleft$ is continuous, and $\boxed{\mathcal{L}_\psi^* = P_\psi} : C(X)^* \circlearrowleft$ is as well.

Furthermore, the RPF operator $\mathcal{L}_\psi$ has the following significance: let ν be ψ-conformal and $h : X \to \mathbb{R}^+$ be a continuous density function. How close is $h\nu$ to being f-invariant? $\forall\, g \in C(X)$, we have

$$\int (g \circ f)(x) h(x) \, d\nu(x) = \int (g\circ f)(x) h(x) \, d(\mathcal{L}_\psi^* \nu)(x)$$

$$= \int \mathcal{L}_\psi [(g \circ f) \cdot h](x) \, d\nu(x)$$

$$= \int \sum_{fy=x} (g\circ f)(y) \psi(y) h(y) \, d\nu(x)$$

$$= \int g(x) (\mathcal{L}_\psi h)(x) \, d\nu(x)$$

$$\therefore \quad f_* : h\nu \longmapsto (\mathcal{L}_\psi h)\nu$$

$\boxed{\text{Fact}}$ If $\log \psi$ is Hölder then $\exists!\ h \in C(X)$ s.t. $\mathcal{L}_\psi h = h$. Then $\mu = h\nu$ is an invariant Gibbs measure.

* For general $\psi \in C(X)$, cannot simply apply a fixed pt theorem – the above requires Arzela-Ascoli to get compactness from equicontinuity.

$$* \ \mathcal{L}_\psi h = h \implies \Big(\mathcal{L}_{\psi \cdot \frac{h}{h\circ f}} 1\Big)(x) = \sum_{fy=x} \psi(y) \frac{h(y)}{h(f(y))} = 1$$

* For general h, $\int g \, d\big(\mathcal{L}^*_{\psi \frac{h}{h\circ f}} (h\nu)\big) = \int \big(\mathcal{L}_{\psi \frac{h}{h\circ f}} g\big)(x) h(x) \, d\nu(x)$

$$= \int \sum_{fy=x} \psi(y) h(y) g(y) \, d\nu(x) = \int hg \, d(\mathcal{L}_\psi^* \nu) = \int g \, d(h\nu)$$

$\therefore$ $h\nu$ is $\psi \frac{h}{h\circ f}$ - conformal.

8 Markov chains & two-step potentials. $X = \Sigma_2^+$

Return to example: $\psi = e^{\varphi}$, look for eigenvectors of $\mathcal{L}_\psi, \mathcal{L}_\psi^*$ (common eigenvalue λ is $e^{P(\varphi)}$). Guess: ν & h will also be two-step (Markov). Say ν is given by (v, P), with

$\nu([x_1 \cdots x_n]) = v_{x_1} P_{x_1 x_2} \cdots P_{x_{n-1} x_n}$. Then

$$\Psi = \begin{bmatrix} e^{\varphi_{00}} & e^{\varphi_{01}} \\ e^{\varphi_{10}} & e^{\varphi_{11}} \end{bmatrix}$$

$$v_i = \nu([i]) = \tfrac{1}{\lambda}(\mathcal{L}_\psi^* \nu)([i]) = \tfrac{1}{\lambda}(\mathcal{P}_\psi \nu)([i])$$
$$= \tfrac{1}{\lambda}\int_{f[i]} e^{\varphi(f^{-1}(x))}\, d\nu(x) = \tfrac{1}{\lambda}\sum_j e^{\varphi_{ij}} v_j = \tfrac{1}{\lambda}(\Psi v)_i$$

so v is given by $\boxed{\Psi v = \lambda v}$. Furthermore,

$$P_{ij} = \nu([ij])/v_i = \frac{1}{v_i}\frac{1}{\lambda}(\mathcal{L}_\psi^* \nu)([ij])$$
$$= \frac{1}{v_i}\frac{1}{\lambda}\int_{f([ij])} e^{\varphi(f^{-1}(x))}\, d\nu(x) = \boxed{\frac{\psi_{ij}\, v_j}{\lambda v_i}}$$

$$\therefore\ \nu([x_1 \cdots x_n]) = \lambda^{-n} e^{S_{n-1}\varphi(x)} v_{x_n}$$

Finally, $h(x) = h(x_1, x_2) = h_{x_1 x_2} \Rightarrow \mathcal{L}_\varphi h(x) = \sum_i \psi(ix) h(ix) = \sum_i \psi_{ix_1} h_{ix_1}$

We want $\lambda h_{x_1 x_2} = \sum_i \psi_{ix_1} h_{ix_1}$ $\quad\therefore$ h only depends on x_1

$$\therefore\ \lambda h_j = \sum_i h_i \psi_{ij} \qquad \therefore\ \boxed{h\Psi = \lambda h}$$

In the case where $\varphi_{ij} = 0$ or $-\infty$, so $\psi_{ij} = 1$ or 0, this reduces to the standard construction of Parry measure, the mme for Σ_Ψ^+.