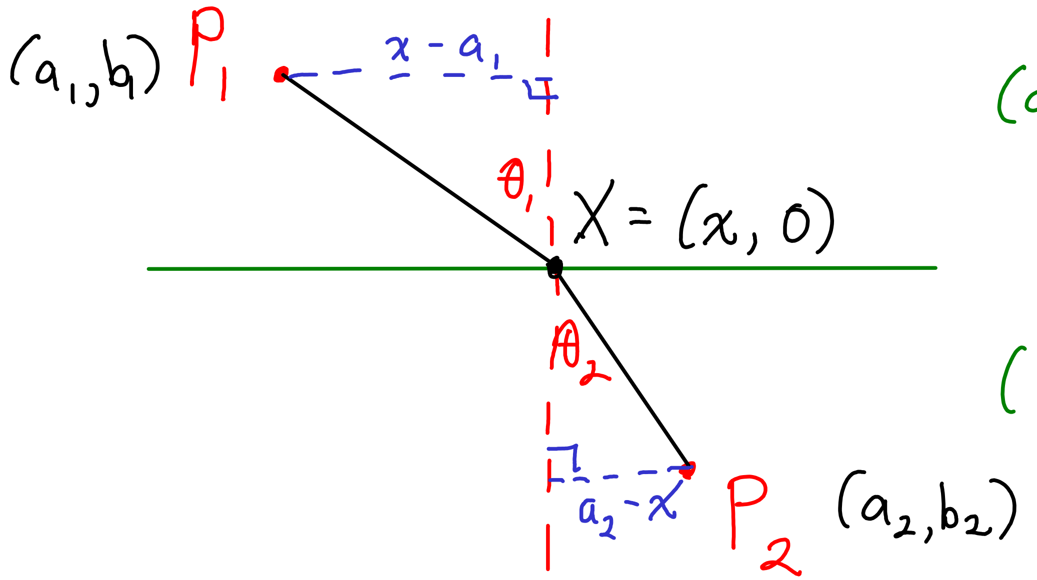
① Snell's law ② Refraction in a water drop ③ Rainbow

① Fermat's principle: light follows the path of least time



(air) velocity v_1

(water) velocity v_2

Index of refraction

$n_i = \frac{1}{v_i}$

$T(x)$ = time from P_1 to X to P_2

Minimize T via calculus. ← find x s.t. $\frac{dT}{dx} = 0$

$$T(x) = \frac{|P_1X|}{v_1} + \frac{|XP_2|}{v_2} = n_1 |P_1X| + n_2 |XP_2|$$

$$T(x) = n_1 \sqrt{(x-a_1)^2 + b_1^2} + n_2 \sqrt{(x-a_2)^2 + b_2^2}$$

$$\frac{dT}{dx} = \frac{n_1 (x-a_1)}{\sqrt{(x-a_1)^2 + b_1^2}} + \frac{n_2 (x-a_2)}{\sqrt{(x-a_2)^2 + b_2^2}} = n_1 \frac{(x-a_1)}{|P_1X|} + n_2 \frac{(x-a_2)}{|XP_2|} = n_1 \sin\theta_1 - n_2 \sin\theta_2$$

This is $= 0$ when

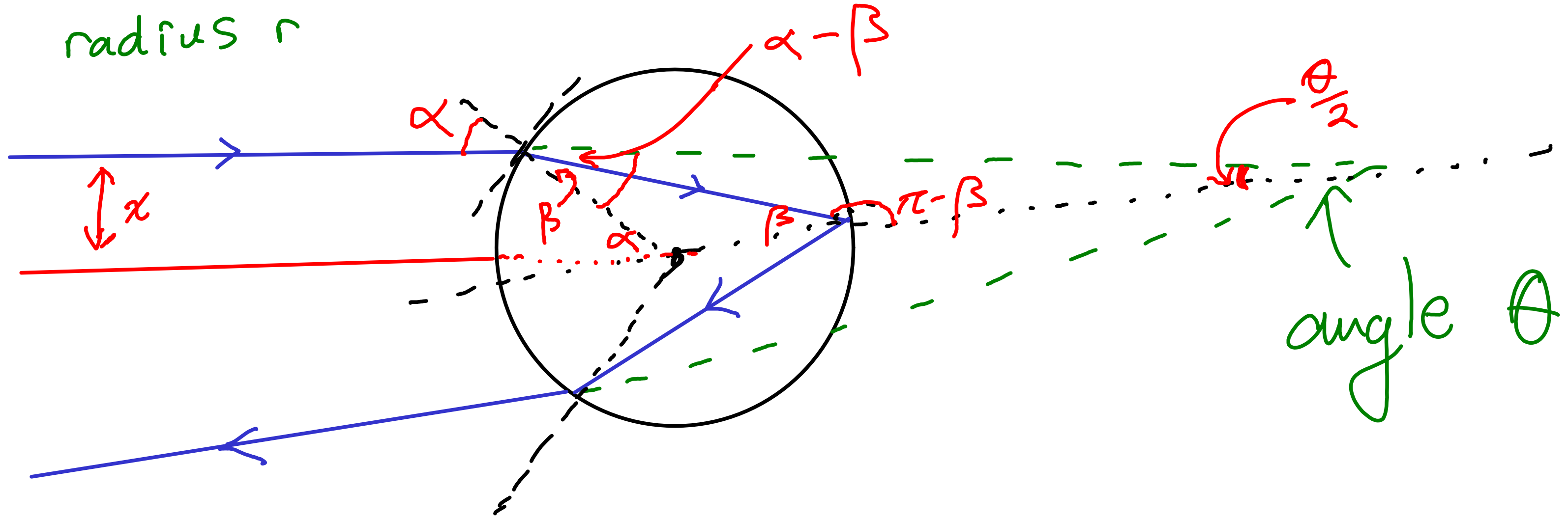
$$\boxed{n_1 \sin\theta_1 = n_2 \sin\theta_2}$$

Snell's Law

For air & water,

$$n = \frac{n_2}{n_1} \approx 1.33 \approx \frac{4}{3}$$

(2)



Plot $\theta(x)$.

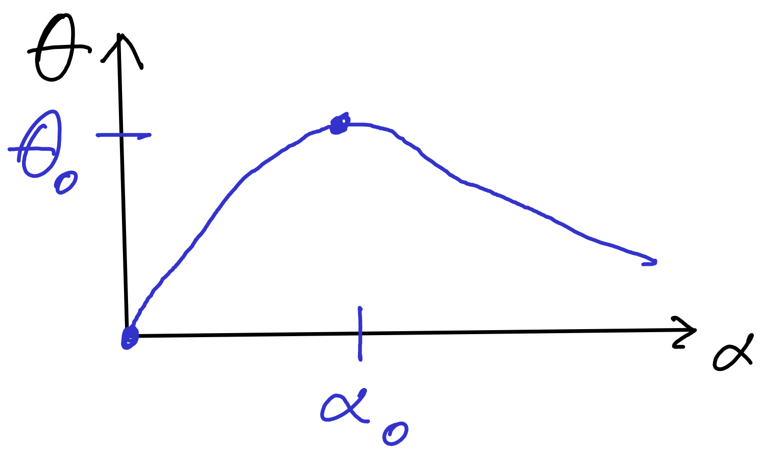
$$\frac{\theta}{2} + (\alpha - \beta) + (\pi - \beta) = \pi$$

$$\frac{\theta}{2} + \alpha - 2\beta = 0 \Rightarrow \boxed{\theta = 4\beta - 2\alpha}$$

Snell's Law: $\sin\alpha = n \sin\beta$

$\sin\alpha = \frac{x}{r}$ $\quad \alpha = \arcsin\frac{x}{r}$

$\sin\beta = \frac{\sin\alpha}{n} = \frac{x}{nr}$



$$\theta = 4\arcsin\left(\frac{x}{nr}\right) - 2\arcsin\left(\frac{x}{r}\right) = 4\arcsin\left(\frac{\sin\alpha}{n}\right) - 2\alpha$$

Find sign of $\frac{d\theta}{d\alpha}$ to sketch graph.

$$\frac{d\theta}{d\alpha} = \frac{\frac{4\cos\alpha}{n}}{\sqrt{1 - \frac{\sin^2\alpha}{n^2}}} - 2 = \frac{4\cos\alpha}{\sqrt{n^2 - \sin^2\alpha}} - 2$$

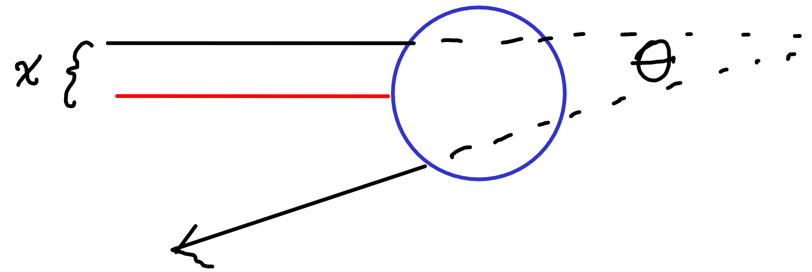
$$\frac{d\theta}{d\alpha} = 0 \Leftrightarrow 2\cos\alpha = \sqrt{n^2 - \sin^2\alpha}$$

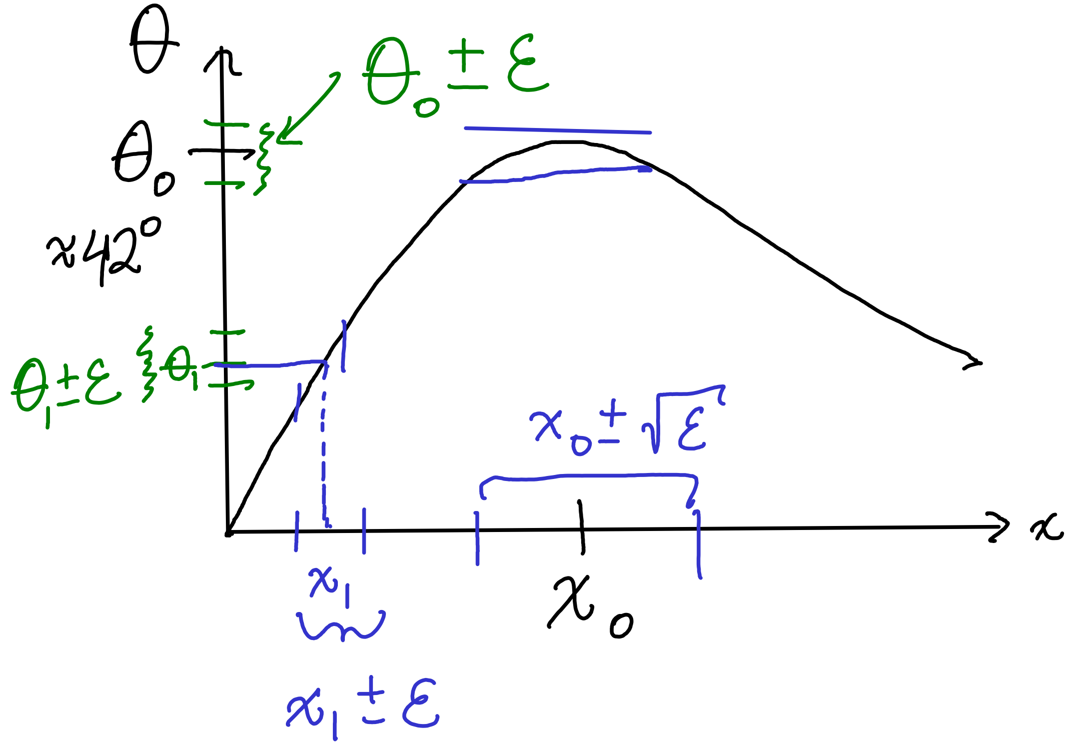
$$\Leftrightarrow 4\cos^2\alpha = n^2 - \sin^2\alpha$$

$$= n^2 - 1 + \cos^2\alpha$$

$$\frac{d\theta}{d\alpha} = 0 \Leftrightarrow 3\cos^2\alpha = n^2 - 1 \approx \frac{16}{9} - 1 = \frac{7}{9} \Leftrightarrow \cos^2\alpha \approx \frac{7}{27} \Leftrightarrow \sin^2\alpha \approx \frac{20}{27} \Rightarrow \theta \approx 42^\circ$$

③ Rainbows

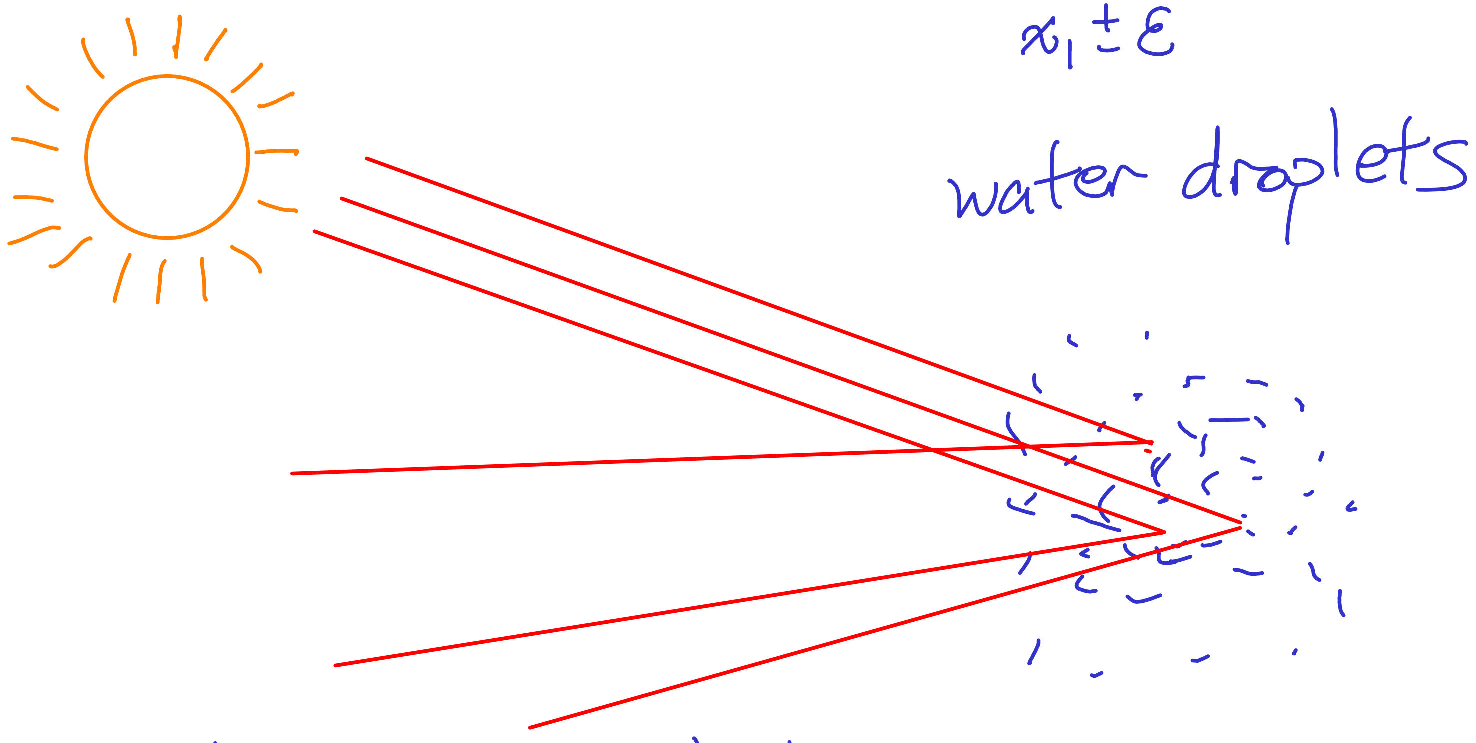




Which outgoing angles occur most often?

Near a critical pt, $\theta(x)$ looks like a quadratic.

Elsewhere, it looks linear.



$\theta_1 \pm \varepsilon$ is produced by a range of x-values with size proportional to ε.

$\theta_0 \pm \varepsilon$ " " " " " " " $\sqrt{\varepsilon}$!

For ε small, $\sqrt{\varepsilon}$ is <u>much</u> larger than ε.

(e.g. $\varepsilon = 10^{-6}$, $\sqrt{\varepsilon} = 10^{-3}$)

θ_0 depends on n, which varies a little with wavelength of light, which produces the rainbow.