# SRB measures for non-uniformly hyperbolic systems

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# Physically meaningful invariant measures

- $M$ a compact Riemannian manifold
- $f\colon M \to M$ a $C^{1+\varepsilon}$ local diffeomorphism
- $\mathcal{M}$ the space of Borel measures on $M$
- $\mathcal{M}(f) = \{\mu \in \mathcal{M} \mid \mu \text{ is } f\text{-invariant}\}$

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Birkhoff ergodic theorem. If $\mu \in \mathcal{M}(f)$ is ergodic then it describes the statistics of $\mu$-a.e. trajectory of $f$: for every integrable $\varphi$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(f^k(x)) = \int \varphi \, d\mu$$

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To be "physically meaningful", a measure should describe the statistics of *Lebesgue*-a.e. trajectory.

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"absolutely continuous" $\rightsquigarrow$ "a.c. *on unstable manifolds*"

$\mu \in \mathcal{M}(f)$ is an SRB measure if

1. all Lyapunov exponents non-zero;
2. $\mu$ has a.c. conditional measures on unstable manifolds.

SRB measures are physically meaningful. Goal: Prove existence of an SRB measure.

# Uniform geometric structure

SRB measures are known to exist in the following settings.

- Uniformly hyperbolic $f$ (Sinai, Ruelle, Bowen)
- Partially hyperbolic $f$ with positive/negative central exponents (Alves–Bonatti–Viana, Burns–Dolgopyat–Pesin–Pollicott)

Key tool is a dominated splitting $T_x M = E^s(x) \oplus E^u(x)$.

1. $E^s$, $E^u$ depend continuously on $x$.
2. $\measuredangle(E^s, E^u)$ is bounded away from 0.

Both conditions fail for non-uniformly hyperbolic $f$.

# Non-uniformly hyperbolic maps

The Hénon maps $f_{a,b}(x, y) = (y + 1 - ax^2, bx)$ are a perturbation of the family of logistic maps $g_a(x) = 1 - ax^2$.

1. $g_a$ has an absolutely continuous invariant measure for “many” values of $a$. (Jakobson)
2. For $b$ small, $f_{a,b}$ has an SRB measure for “many” values of $a$. (Benedicks–Carleson, Benedicks–Young)
3. Similar results for “rank one attractors” – small perturbations of one-dimensional maps with non-recurrent critical points. (Wang–Young)

Genuine non-uniform hyperbolicity, but only one unstable direction, and stable direction must be strongly contracting.

# Constructing invariant measures

- $f$ acts on $\mathcal{M}$ by $f_*\colon m \mapsto m \circ f^{-1}$.
- Fixed points of $f_*$ are invariant measures.
- Césaro averages + weak* compactness $\Rightarrow$ invariant measures:

$$\mu_n = \frac{1}{n}\sum_{k=0}^{n-1} f_*^k m \qquad\qquad \mu_{n_j} \to \mu \in \mathcal{M}(f)$$

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Idea: $m =$ volume $\Rightarrow \mu$ is an SRB measure.

$$H = \{x \in M \mid \text{all Lyapunov exponents non-zero at } x\}$$

$$\mathcal{S} = \{\nu \in \mathcal{M} \mid \nu(H) = 1, \nu \text{ a.c. on unstable manifolds}\}$$

- $\mathcal{S} \cap \mathcal{M}(f) = \{\text{SRB measures}\}$
- $\mathcal{S}$ is $f_*$-invariant, so $\mu_n \in \mathcal{S}$ for all $n$.
- $\mathcal{S}$ is *not* compact. So why should $\mu$ be in $\mathcal{S}$?

# Non-uniform hyperbolicity in $\mathcal{M}$

Theme in NUH: choose between invariance and compactness.

Replace unstable manifolds with $n$-admissible manifolds $V$:

$d(f^{-k}(x), f^{-k}(y)) \leq Ce^{-\lambda k} d(x, y)$ for all $0 \leq k \leq n$ and $x, y \in V$.

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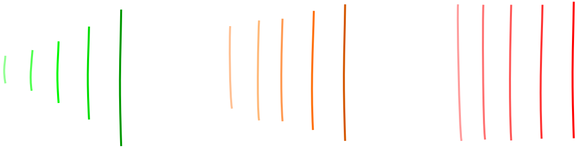
$\mathcal{S}_n = \{\nu$ supp. on and a.c. on $n$-admissible manifolds, $\nu(H) = 1\}$.
This set of measures has various non-uniformities.

1. Value of $C, \lambda$ in definition of $n$-admissibility.



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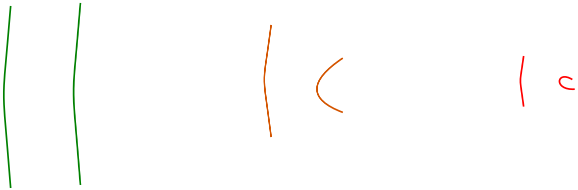

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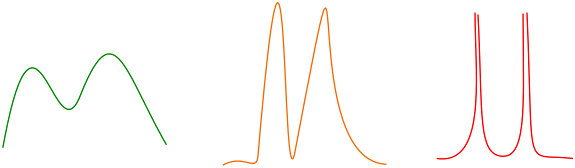
3. $\|\rho\|$, where $\rho$ is density wrt. leaf volume.



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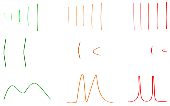

Given $K > 0$, let $\mathcal{S}_n(K)$ be the set of measures for which these non-uniformities are all controlled by $K$.

$$\text{large } K \Rightarrow \text{worse non-uniformity}$$

$\mathcal{S}_n(K)$ is compact, but not $f_*$-invariant.

# Conditions for existence of an SRB measure

- $M$ be a compact Riemannian manifold, $U \subset M$ open, $f: U \to M$ a local diffeomorphism with $\overline{f(U)} \subset U$.
- Let $\mu_n$ be a sequence of measures whose limit measures are all invariant.
- Fix $K > 0$, write $\mu_n = \nu_n + \zeta_n$, where $\nu_n \in \mathcal{S}_n(K)$.

**Theorem (C.–Dolgopyat–Pesin, 2010)**

*If $\overline{\lim}_{n\to\infty} \|\nu_n\| > 0$, then some limit measure of $\{\mu_n\}$ has an ergodic component that is an SRB measure for $f$.*

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The question now becomes: How do we obtain recurrence to the set $S_n(K)$?

# Coordinates in $TM$

We use local coordinates to write the map $f$ along a trajectory as a sequence of local diffeomorphisms.

- $\{f^n(x) \mid n \geq 0\}$ is a trajectory of $f$
- $U_n \subset T_{f^n(x)}M$ is a neighbourhood of 0 small enough so that the exponential map $\exp_{f^n(x)}: U_n \to M$ is injective
- $f_n: U_n \to \mathbb{R}^d = T_{f^{n+1}(x)}M$ is the map $f$ in local coordinates

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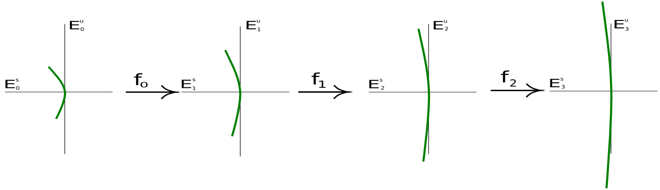
Suppose $\mathbb{R}^d = T_{f^n(x)}M$ has an invariant decomposition $E_n^u \oplus E_n^s$ with asymptotic expansion (contraction) along $E_n^u$ ($E_n^s$).

$$
\begin{aligned}
Df_n(0) &= A_n \oplus B_n \\
f_n &= Df_n(0) + s_n \\
f_n(v, w) &= (A_n v + g_n(v, w), B_n w + h_n(v, w))
\end{aligned}
$$

# Controlling hyperbolicity and regularity

$$\mathbb{R}^d = E_n^u \oplus E_n^s \qquad f_n = (A_n \oplus B_n) + s_n$$

Start with an admissible manifold $V_0$ tangent to $E_0^u$ at 0, push it forward and define an invariant sequence of admissible manifolds by $V_{n+1} = f_n(V_n)$.



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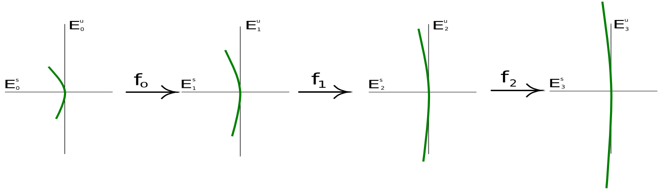
$$V_n = \operatorname{graph} \psi_n = \{v + \psi_n(v)\} \qquad \psi_n \colon B(E_n^u, r_n) \to E_n^s$$

Need to control the size $r_n$ and the regularity $\|D\psi_n\|$, $|\psi_n|_\varepsilon$.



# Controlling hyperbolicity and regularity

Consider the following quantities:

$$\lambda_n^u = \log(\|A_n^{-1}\|^{-1}) \qquad \lambda_n^s = \log\|B_n\|$$
$$\alpha_n = \measuredangle(E_n^u, E_n^s) \qquad C_n = \|s_n\|_{C^{1+\varepsilon}}$$

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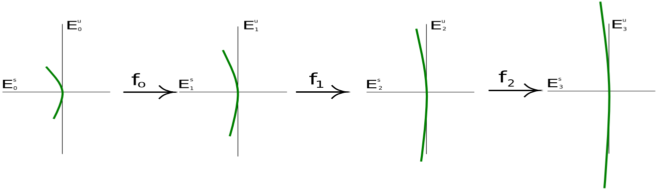
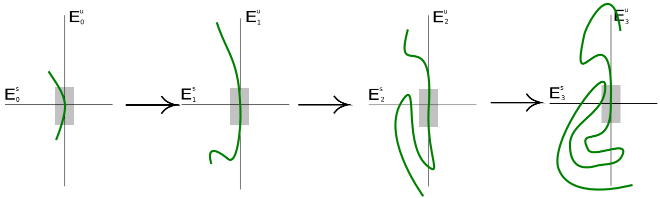


# Classical Hadamard–Perron results

Uniform case: Constants such that

- $\lambda_n^s \leq \bar{\lambda}^s < 0 < \bar{\lambda}^u < \lambda_n^u$
- $\alpha_n \geq \bar{\alpha} > 0$
- $C_n \leq \bar{C} < \infty$

Then $V_n$ has uniformly large size: $r_n \geq \bar{r} > 0$.



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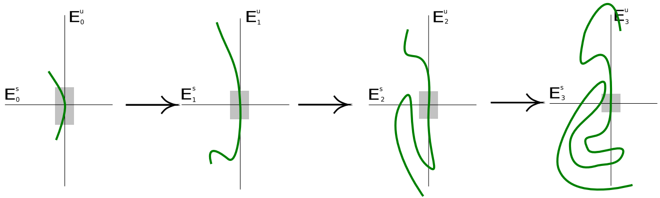
Non-uniform case: $\lambda_n^s, \lambda_n^u, \alpha_n$ still uniform, but $C_n$ not.

$$C_n \text{ grows slowly} \Rightarrow r_n \text{ decays slowly}$$



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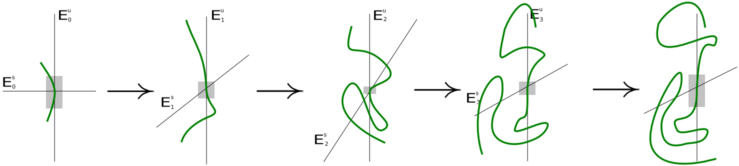
We want to consider the case where

- $\lambda_n^s < 0 < \lambda_n^u$ may fail (may even have $\lambda_n^u < \lambda_n^s$)
- $\alpha_n$ may become arbitrarily small
- $C_n$ may become arbitrarily large (no control on speed)

# Usable hyperbolicity

In order to define $\psi_{n+1}$ implicitly, we need control of the regularity of $\psi_n$. Control $\|D\psi_n\|$ and $|D\psi_n|_\varepsilon$ by decreasing $r_n$ if necessary. So how do we guarantee that $r_n$ becomes "large" again?



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$$\beta_n = C_n (\sin \alpha_{n+1})^{-1}$$

Fix a threshold value $\bar{\beta}$ and define the usable hyperbolicity:

$$\lambda_n = \begin{cases} \min\left(\lambda_n^u, \lambda_n^u + \frac{1}{\varepsilon}(\lambda_n^u - \lambda_n^s)\right) & \text{if } \beta_n \le \bar{\beta}, \\ \min\left(\lambda_n^u, \lambda_n^u + \frac{1}{\varepsilon}(\lambda_n^u - \lambda_n^s), \frac{1}{\varepsilon}\log\frac{\beta_n}{\beta_{n+1}}\right) & \text{if } \beta_n > \bar{\beta}. \end{cases}$$

# A Hadamard–Perron theorem

Write $F_n = f_{n-1} \circ \cdots \circ f_1 \circ f_0 \colon U_0 \to \mathbb{R}^d = T_{f^n(x)}M$. Let $V_0 \subset \mathbb{R}^d$ be a $C^{1+\varepsilon}$ manifold tangent to $E_0^u$ at 0, and let $V_n(r)$ be the connected component of $F_n(V_0) \cap (B(E_n^u, r) \times E_n^s)$ containing 0.

**Theorem (C.–Dolgopyat–Pesin, 2010)**

*Suppose $\bar\beta$ and $\bar\chi > 0$ are such that $\underline{\lim}_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \lambda_k > \bar\chi > 0$. Then there exist constants $\bar\gamma, \bar\kappa, \bar r > 0$ and a set $\Gamma \subset \mathbb{N}$ with positive lower asymptotic frequency such that for every $n \in \Gamma$,*

1. *$V_n(\bar r)$ is the graph of a $C^{1+\varepsilon}$ function $\psi_n \colon B_{E_n^u}(\bar r) \to E_n^s$ satisfying $\|D\psi_n\| \le \bar\gamma$ and $|D\psi_n|_\varepsilon \le \bar\kappa$;*
2. *if $F_n(x), F_n(y) \in V_n(\bar r)$, then for every $0 \le k \le n$, $\|F_n(x) - F_n(y)\| \ge e^{k\bar\chi} \|F_{n-k}(x) - F_{n-k}(y)\|$.*

# Usable hyperbolicity (again)

Given a measurable invariant decomposition $T_xM = E^s(x) \oplus E^u(x)$ for $x \in A \subset U$, define $\lambda^u, \lambda^s \colon A \to \mathbb{R}$ by

$$\lambda^u(x) = \inf\{\log \|Df(v)\| \mid v \in E^u(x), \|v\| = 1\},$$
$$\lambda^s(x) = \sup\{\log \|Df(v)\| \mid v \in E^s(x), \|v\| = 1\}.$$

Let $\alpha(x)$ be the angle between $E^s(x)$ and $E^u(x)$. Fix $\bar\alpha > 0$ and consider the quantities

$$\zeta(x) = \begin{cases} \frac{1}{\varepsilon} \log \frac{\alpha(f(x))}{\alpha(x)} & \text{if } \alpha(x) < \bar\alpha, \\ +\infty & \text{if } \alpha(x) \geq \bar\alpha. \end{cases}$$

$$\lambda(x) = \min\left\{\lambda^u(x),\ \lambda^u(x) + \frac{1}{\varepsilon}(\lambda^u(x) - \lambda^s(x)),\ \zeta(x)\right\}$$

# An existence result

Consider points with positive asymptotic usable hyperbolicity:

$$S = \left\{ x \in A \,\middle|\, \varliminf_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \lambda(f^k(x)) > 0 \text{ and } \varlimsup_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \lambda^s(f^k(x)) < 0 \right\}.$$

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*If there exists $\bar{\alpha} > 0$ such that $\mathrm{Leb}\, S > 0$, then $f$ has a hyperbolic SRB measure supported on $\Lambda$.*

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**Theorem (C.–Dolgopyat–Pesin, 2010)**

*Fix $x \in U$. Let $V(x)$ be an embedded submanifold such that $T_x V(x) \subset K^u(x)$, and let $m_V$ be leaf volume on $V(x)$. Suppose that there exists $\bar{\alpha} > 0$ such that $\varliminf_{r\to 0} m_V(S \cap B(x,r)) > 0$. Then $f$ has a hyperbolic SRB measure supported on $\Lambda$.*