Thermodynamics with small obstructions

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October 31, 2011

Joint work with Daniel J. Thompson (Penn State)

The talk in one slide

specification $\Rightarrow$ intrinsic ergodicity (unique MME)

Theorem (C.–Thompson)

If all obstructions to specification have small entropy, then the MME is unique.

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specification + Bowen property $\Rightarrow$ unique equilibrium state

Theorem (C.–Thompson)

If all obstructions to specification and the Bowen property have small pressure, then the equilibrium state is unique.

Topological pressure

Topological dynamical system:

- X a compact metric space, $f\colon X \to X$ continuous
- $\mathcal{M} = \{\text{Borel } f\text{-invariant probability measures on } X\}$

Variational principle 1: $h_{\text{top}}(X, f) = \sup_{\mu \in \mathcal{M}} h_\mu(f)$

Variational principle 2: $P(\varphi) = \sup_{\mu \in \mathcal{M}} \left(h_\mu(f) + \int \varphi \, d\mu\right)$

Maximum achieved by MME/equilibrium state

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Example

$$X = \Sigma_2^+ = \{0,1\}^{\mathbb{N}} \qquad p, q \in \mathbb{R} \qquad \varphi(x) = p\mathbf{1}_{[0]} + q\mathbf{1}_{[1]}$$

Then $P(\varphi) = \log(e^p + e^q)$ and the unique equilibrium state is $(\alpha, 1-\alpha)$-Bernoulli, where $\alpha = \frac{e^p}{e^p + e^q}$.

When is there a unique equilibrium state?

Thermodynamic formalism

Topological pressure $P(\varphi) = \sup_\mu (h_\mu(f) + \int \varphi \, d\mu)$:

- supremum of affine functions $\Rightarrow$ convex function $C(X) \to \mathbb{R}$;
- equilibrium states are tangent functionals in $C(X)^*$.

If entropy map $\mu \mapsto h_\mu(f)$ is upper semi-continuous, then:

- so is $\mu \mapsto (h_\mu(f) + \int \varphi \, d\mu)$ $\Rightarrow$ *existence*;
- *uniqueness* $\Leftrightarrow$ (Gâteaux) differentiability of P;
- convex $\Rightarrow$ differentiable on residual set $\Rightarrow$ *uniqueness*.

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This says nothing about **which** potentials admit unique equilibrium states, or what the ergodic properties of those states are.

General principle: Uniform mixing of X and regularity of φ should imply uniqueness and strong ergodic properties.

Thermodynamics for shift spaces

Focus on shift spaces (subshifts):

- $X \subset \Sigma_p^+$ closed and σ-invariant, NOT NECESSARILY AN SFT
- language of X: $\mathcal{L} = \mathcal{L}(X) = \{x_1 \cdots x_n \mid x \in X, n \geq 1\}$
- words of length n: $\mathcal{L}_n = \{w \in \mathcal{L} \mid |w| = n\}$
- entropy: $h_{\text{top}}(X, \sigma) = \lim_{n\to\infty} \frac{1}{n} \log \#\mathcal{L}_n$
- pressure: $P(\varphi) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{w\in\mathcal{L}_n} e^{\sup_{x\in[w]} S_n\varphi(x)}$
- $S_n\varphi(x) = \varphi(x) + \varphi(\sigma x) + \cdots + \varphi(\sigma^{n-1}x)$

expansive $\Rightarrow$ entropy map $\mathcal{M} \to \mathbb{R}$ upper semi-continuous

Existence guaranteed: the real question is uniqueness.

Beyond transitivity, what do we need?

Uniqueness may fail

Example

Let $X \subset \Sigma_5^+ = \{0, 1, 2, 1, 2\}^{\mathbb{N}}$ be the shift whose language $\mathcal{L}$ contains $v0^n w$ and $w0^n v$ if and only if $n \geq 2\max(|v|, |w|)$.

- (X, σ) is topologically transitive (indeed, mixing)
- $h_{\text{top}}(X, \sigma) = \log 2$
- 2 measures of maximal entropy:

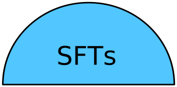
$$\nu = (\tfrac{1}{2}, \tfrac{1}{2})\text{-Bernoulli on } \{1, 2\}^{\mathbb{N}},$$
$$\mu = (\tfrac{1}{2}, \tfrac{1}{2})\text{-Bernoulli on } \{1, 2\}^{\mathbb{N}}.$$

Uniqueness of an MME can fail for transitive shifts.

Classes of intrinsically ergodic shifts

The following are intrinsically ergodic (unique MME):

- Irreducible subshifts of finite type (Parry 1964)



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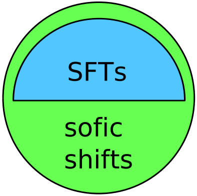

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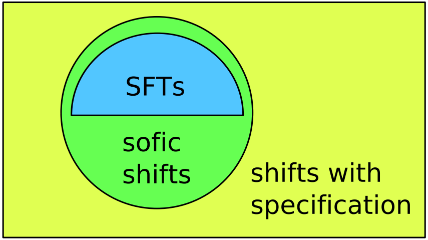

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The following are intrinsically ergodic (unique MME):

- Irreducible subshifts of finite type (Parry 1964)
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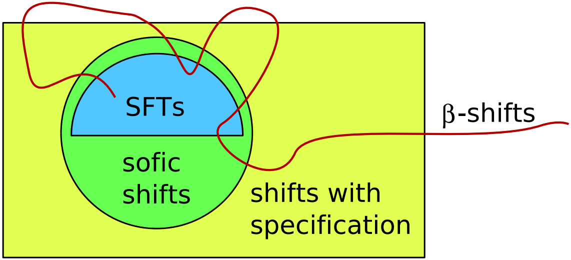
- Shifts with specification (Bowen 1974)
- β-shifts (Walters 1978, Hofbauer 1979)



The motivating question

Intrinsic ergodicity is not necessarily preserved by factors.

- $X \subset \{0, 1, 2, 1, 2\}^{\mathbb{N}}$ as before
- $Y \subset \Sigma_6^+ = \{0, 1, 2, 1, 2, 3\}^{\mathbb{N}}$ by similar rule
- X is a factor of Y; Y is intrinsically ergodic; X is not

Specification is preserved by factors, so intrinsic ergodicity survives.

What about β-shifts? (Klaus Thomsen, Mike Boyle)

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What about β-shifts? (Klaus Thomsen, Mike Boyle)

Theorem (C.–Thompson 2010)

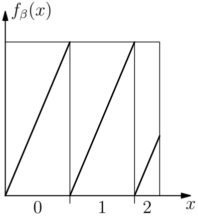
(X, σ) a subshift factor of a β-shift
$\Rightarrow$ all obstructions to specification have zero entropy
$\Rightarrow (X, \sigma)$ is intrinsically ergodic

β-shifts

For $\beta > 1$, Σ_β is the coding space for the map

$$f_\beta \colon [0,1] \to [0,1], \qquad x \mapsto \beta x \pmod 1$$

$1_\beta = a_1 a_2 \cdots$, where $1 = \sum_{n=1}^{\infty} a_n \beta^{-n}$



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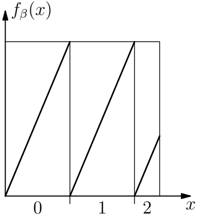
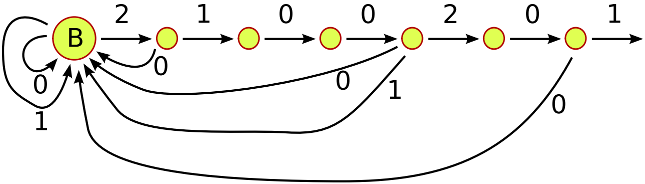


$1_\beta = a_1 a_2 \cdots$, where $1 = \sum_{n=1}^{\infty} a_n \beta^{-n}$

Fact: $x \in \Sigma_\beta$ iff x labels a walk starting at **B** on the graph shown.
(Here $1_\beta = 2100201\dots$)



Specification

Topological transitivity $\Rightarrow$ for every $w_1, \dots, w_m \in \mathcal{L}$ $\exists$ $z_i \in \mathcal{L}$ for which the concatenated word $w_1 z_1 w_2 z_2 \cdots z_{m-1} w_m$ is in $\mathcal{L}$.

Definition

X has specification if $\exists$ $t \in \mathbb{N}$ such that z_i can always be chosen to have length t, independently of w_i.

- Mixing SFTs and sofic shifts have specification.
- Σ_β does not have specification if 1_β contains arbitrarily long strings of 0's. (Happens for Leb-a.e. $\beta > 1$.)

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The only obstruction to specification is the tail of the sequence 1_β.

Key idea: zero entropy obstructions are invisible to MMEs

Obstructions to specification

What do we mean by "obstruction"?

Definition

$\mathcal{G} \subset \mathcal{L}$ is a core for specification if

- Every $\mathcal{G}(M) := \{vw \in \mathcal{L} \mid v \in \mathcal{G}, |w| \leq M\}$ has specification

$\mathcal{C} \subset \mathcal{L}$ contains all obstructions to specification if $\exists$ core $\mathcal{G}$ s.t.

- $\mathcal{L} = \mathcal{G}\mathcal{C} := \{vw \in \mathcal{L} \mid v \in \mathcal{G}, w \in \mathcal{C}\}$

We can glue words (orbit segments) together, provided we are allowed to remove an obstructing piece from the end of each word.

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Example

For the β-shift, take $\mathcal{C} = \{\text{prefixes of } 1_\beta\}$. ▸ GRAPH

- $\mathcal{C} =$ words whose path never returns to **B** (cusp excursions)
- $\mathcal{G} =$ words whose path begins and ends at **B**

Small obstructions

Theorem (C.–Thompson 2010)

If X is a shift space, $\mathcal{C}$ contains all obstructions to specification, and $h(\mathcal{C}) < h_{\text{top}}(X, \sigma)$, then (X, σ) is intrinsically ergodic.

Remark: If the shadowing orbits can be taken to be periodic, then the unique MME is the limit of the measures $\mu_n = \delta_{\text{Per}(n)}$.

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Proposition

$\pi : X \to Y$ a factor map, $\mathcal{C} \subset \mathcal{L}(X)$ contains all obstructions $\Rightarrow$

- *$\pi(\mathcal{C}) \subset \mathcal{L}(Y)$ also contains all obstructions*
- *$h(\pi(\mathcal{C})) \leq h(\mathcal{C})$*

Coded systems

A shift space X is coded if its language $\mathcal{L}$ is freely generated by a countable set of generators $\{w_n\}_{n\in\mathbb{N}} \subset \mathcal{L}$.

$$\mathcal{L} = \{\text{all subwords of } w_{n_1} w_{n_2} \cdots w_{n_k} \mid n_i \in \mathbb{N}\}$$

Canonical decomposition $\mathcal{L} = \mathcal{C}^s \mathcal{G} \mathcal{C}^p$ s.t. $\mathcal{G}(M)$ has specification:

$$\mathcal{G} = \{w_{n_1} w_{n_2} \cdots w_{n_k} \mid n_i \in \mathbb{N}\}$$
$$\mathcal{C}^s = \{\text{suffixes of } w_n \mid n \in \mathbb{N}\} \qquad \mathcal{C}^p = \{\text{prefixes of } w_n\}$$

Obstruction: prefixes and suffixes of generators

Let $\hat{h} = h(\{\text{prefixes and suffixes of generators}\})$.

- $\hat{h} < h_{\text{top}}(X,\sigma) \Rightarrow (X,\sigma)$ is intrinsically ergodic
- $\hat{h} = 0 \Rightarrow$ every subshift factor of (X,σ) is intrinsically ergodic

The Bowen property

Unique equilibrium states? Specification is not enough

Example (Manneville–Pomeau map)

- $f(x) = x + x^{1+\varepsilon} \pmod 1$, $\varphi_t(x) = -t \log f'(x)$
- $t \neq 1$: unique equilibrium state, fully supported/atomic
- $t = 1$: two equilibrium states

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Given $X \subset \Sigma_p^+$ and $\mathcal{D} \subset \mathcal{L}(X)$, the *n*th variation of ϕ on $\mathcal{D}$ is

$$V_n(\mathcal{D}, \phi) = \sup_{w \in \mathcal{D}_n} \sup_{x,y \in [w]} |\phi(x) - \phi(y)|$$

Definition

φ has the Bowen property if $\sup_n V_n(\mathcal{L}, S_n\varphi) < \infty$.

Utility of the Bowen property

Definition of pressure:

$$P(\mathcal{D}, \varphi) = \lim_{n\to\infty} \frac{1}{n} \log \Lambda_n(\mathcal{D}, \varphi)$$

$$\Lambda_n(\mathcal{D}, \varphi) = \sum_{w\in\mathcal{D}_n} e^{\sup_{x\in[w]} S_n\varphi(x)}.$$

What happens if we replace x with some other $y \in [w]$?

- Continuity $\Rightarrow$ exponential growth rate preserved
- Bowen property $\Rightarrow$ Λ_n **preserved to within multiplicative constant**

(X uniformly expanding $\Rightarrow$ Hölder continuous functions have the Bowen property.)

Unique equilibrium states

Theorem (Bowen 1974)

Consider a system (X, f) and a potential $\varphi\colon X \to \mathbb{R}$. Suppose

1. *X a compact metric space;*
2. *f a continuous map;*
3. *f is expansive;*
4. *f has specification;*
5. *φ has the Bowen property.*

Then φ has a unique equilibrium state.

Goal: Replace these with non-uniform versions. Expect to get same result provided obstructions to all properties are small.

Uniqueness in the presence of obstructions

Definition

φ has the Bowen property on $\mathcal{G}$ if

- $\sup_n V_n(\mathcal{G}, S_n\varphi) < \infty$

$\mathcal{C} \subset \mathcal{L}$ contains all obstructions to the Bowen property for φ if

- φ is Bowen on some $\mathcal{G} \subset \mathcal{L}$ with $\mathcal{L} = \mathcal{G}\mathcal{C}$.

Theorem (C.–Thompson 2011)

Let X be a shift space and $\varphi \in C(X)$. If $\mathcal{C}$ contains all obstructions to specification and the Bowen property for φ and $P(\mathcal{C}, \varphi) < P(X, \varphi)$, then there is a unique equilibrium state for φ.

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Remarks:

- μ_φ has a weak Gibbs property.
- If $\mathcal{G}(M)$ has (Per)-specification, then $\mu_\varphi = \lim_n \delta^{\varphi}_{\mathrm{Per}(n)}$.

Equilibrium states for β-shifts

Let $(X, f) = (\Sigma_\beta, \sigma)$ be a β-shift.

Theorem (Walters 1978)

Every Lipschitz potential φ has a unique equilibrium state.

Theorem (Hofbauer–Keller 1982)

If φ has the Bowen property and $\sup \varphi - \inf \varphi < h_{\text{top}}(X, f)$, then φ has a unique equilibrium state.

Theorem (C.–Thompson 2011)

Every Bowen potential φ has a unique equilibrium state.

Variants of Manneville–Pomeau

Example (Generalisation of Manneville–Pomeau)

- $\gamma > 0 \quad \rightsquigarrow \quad f(x) = x + \gamma x^{1+\varepsilon} \pmod 1$.
- Topologically (semi-)conjugate to Σ_β for some $\beta > 1$.
- For most values of γ, does not have specification.
- $\varphi_t(x) = -t \log f'(x)$ does not have the Bowen property.

Theorem (C.–Thompson 2011)

For $t < 1$ and $\varepsilon \in (0, 1)$, there is a unique equilibrium state for φ_t.

Proof sketch for unique MME (Bowen's proof)

Step 1. Constructible MME μ

Step 2. μ is ergodic and Gibbs.

Step 3. No room for another MME $\nu \perp \mu$.

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$$\mu((\mathcal{D} \cap \mathcal{G}(M))_n) \geq K(M)e^{-nh}\#(\mathcal{D} \cap \mathcal{G}(M))_n \geq K(M)C(M) > 0$$

Bowen potentials on β-shifts

In general, choose $\mathcal{G}$ to deal with two sources of bad behaviour:

1. failure of specification;
2. failure of the Bowen property.

X a β-shift and φ a Bowen potential. Decomposition as before.

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- Need to check that $P(\mathcal{C}, \varphi) < P(X, \varphi)$.

For a β-shift, $\mathcal{C}_n$ contains exactly one word.

$$\varphi = 0 \;\Rightarrow\; P(\mathcal{C}, \varphi) = h(\mathcal{C}) = 0 < h(\mathcal{L}) = P(X, \varphi)$$

For $\varphi \neq 0$, need a new argument that $P(\mathcal{C}, \varphi) < P(X, \varphi)$. Amounts to checking

$$\overline{\lim_{n\to\infty}} \frac{1}{n} S_n\varphi(1_\beta) < P(X, \varphi).$$

Estimating $P(\mathcal{C}, \varphi)$, Part I

- Write $1_\beta = u_1 u_2 u_3 \cdots$, where $u_j = 0^{\ell_j} a_j$ for some $a_j > 0$.
- Can replace any u_j with $\hat{u}_j = 0^{\ell_j} 0$ and get a legal word.
- Let $N = \sum_{j=1}^n (\ell_j + 1) = |u_1 u_2 \cdots u_n|$. Changing k of the first n words u_j to $\hat{u}_j$ gives $\binom{n}{k}$ distinct words $w \in \mathcal{L}_N$.
- For each w, every $x \in [w]$ has $S_N\varphi(x) \geq S_N\varphi(1_\beta) - 5kV$, where $V = \sup_m \sup_{w \in \mathcal{L}_m} \sup_{x,y \in [w]} |S_m\varphi(x) - S_m\varphi(y)|$.
- $\Lambda_N(\mathcal{L}, \varphi) \geq \sum_{k=0}^n \binom{n}{k} e^{S_N\varphi(1_\beta) - 5kV} = e^{S_N\varphi(1_\beta)} (1 + e^{-5V})^n$.
- If 1_β has a positive frequency of non-zero symbols, then $n \geq \delta N$ for some $\delta > 0$, so this suffices.

Estimating $P(\mathcal{C}, \varphi)$, Part II

- If $\frac{n}{N} \to 0$, then $\delta_{1_\beta, n} \to \delta_0$ in weak* topology, so $\frac{1}{n} S_n\varphi(1_\beta) \to \varphi(0)$.
- Fix $1 < \beta' < \beta$ such that $\Sigma_{\beta'}$ is an SFT.
- Then there is a unique equilibrium state for $\varphi|_{\Sigma_{\beta'}}$, which is fully supported, so

$$\varphi(0) < P(\Sigma_{\beta'}, \varphi) \leq P(\Sigma_\beta, \varphi).$$

- If $\underline{\lim} \frac{n}{N} = 0$ and $\overline{\lim} \frac{n}{N} > 0$, need to combine both arguments.