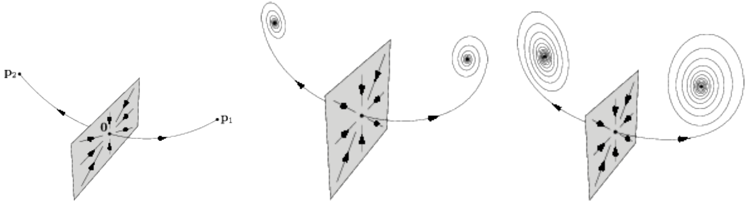
Lorenz system

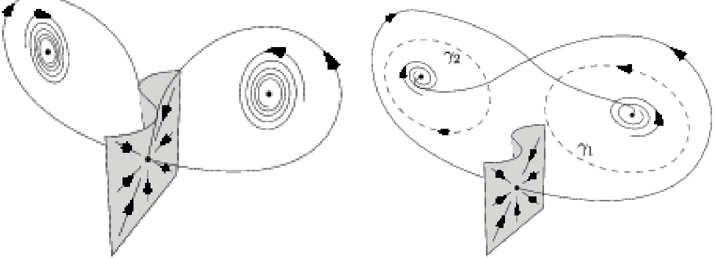
$$\dot{x} = \sigma(y - x)$$
$$\dot{y} = x(\rho - z) - y$$
$$\dot{z} = xy - \beta z$$

Fix $\sigma = 10, \beta = \frac{8}{3}$, let ρ vary.

- $0 < \rho < 1$: only fixed point is 0, globally attracting
- $1 < \rho < \rho_0 := 13.926\ldots$: now 0 has an unstable direction, two other attracting fixed points

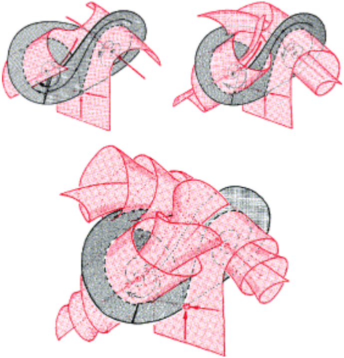


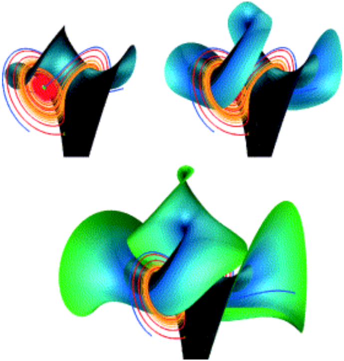
At $\rho = \rho_0$ there is a *homoclinic bifurcation*: the unstable manifold of 0 comes back and approaches 0 through the stable manifold, so it is a homoclinic orbit.



For $\rho > \rho_0$, the unstable curve crosses past the stable manifold (without intersecting it) and approaches the other fixed point. This forces the geometry of the stable manifold to be very complicated.

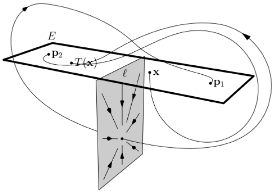
Stable manifold for 0 sometime after the homoclinic bifurcation:

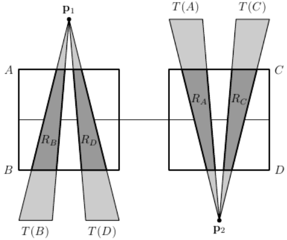




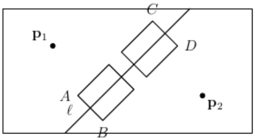
First picture: Abraham and Shaw (1982), hand-drawn
Second picture: Osinga and Krauskopf (2002), computer-generated

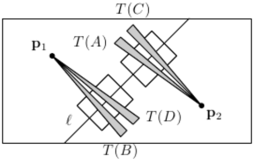
How to understand behaviour of the system? Use a Poincaré section.





There is a **horseshoe**. This leads to **intermittent chaos**.





At $\rho = \rho_1 \approx 24.05$, behaviour changes – p_1, p_2 are contained in the horseshoe, get **chaotic attractor** – *persistent chaos*.