PROBLEM SET 2

Due *5pm Thursday Oct. 18.* You must give complete justification for all answers in order to receive full credit.

1. Let $A, B \in \mathbb{M}_{n\times n}$ commute, and let $V \subset \mathbb{C}^n$ be a generalised eigenspace for A. Show that V is B-invariant. (In fact, one can make the slightly stronger statement that $\ker(A - \lambda I)^k$ is B-invariant for every k.)

2. Let $A \in \mathbb{M}_{4\times 4}$ have eigenvalues $\{-2, -2, -i, i\}$ (counting multiplicity). Write down all possible upper real Jordan canonical forms for A. For each such canonical form, give the general solution to $\dot{x} = Ax$. *(Leave the general solution in the form $M(t)x(0)$, where $M(t)$ is a time-dependent 4×4 matrix and $x(0)$ is the initial condition.)*

3. For each of the following matrices A, find the stable, unstable, and centre subspaces E^s, E^u, and E^c, and use these to sketch the phase portrait of solutions to $\dot{x} = Ax$.

(a) $A = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}$ (b) $A = \begin{pmatrix} 0 & -4 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}$

4. Say that a solution to an ODE is *complicated* if it is bounded for all time (there exists $M > 0$ such that $|x(t)| \leq M$ for all $t \in \mathbb{R}$) and is not an equilibrium or a periodic orbit.

(a) Suppose that $A \in \mathbb{M}_{n\times n}$ has at most one complex conjugate pair of non-real eigenvalues, and show that $\dot{x} = Ax$ has no complicated solutions.

(b) Let $A \in \mathbb{M}_{4\times 4}$ have eigenvalues $\pm i, \pm \omega i$, where $\omega \in \mathbb{R}$ is irrational. Show that the ODE $\dot{x} = Ax$ has a complicated solution.

5. Let $\varphi_t \colon \mathbb{R}^2 \to \mathbb{R}^2$ be the flow corresponding to $\dot{x} = Ax$. Show that φ_t is a linear map on $\mathbb{R}^2$ for each fixed t, which preserves area if and only if $\mathrm{Tr}(A) = 0$, and that in this case the origin is not a sink or a source.

6. For each of the following nonlinear systems $\dot{x} = f(x)$, find all the equilibrium points and classify them as sources, sinks, or saddles.

(a) $f(x_1, x_2) = (-2x_2 + x_1x_2 - 4, 4x_2^2 - x_1^2)$

(b) $f(x_1, x_2) = (4x_1 - 4x_1x_2, 2x_2 - x_1^2 + x_2^2)$

7. Compute the first three Picard iteration terms $x_0(t), x_1(t), x_2(t)$ for the initial value problem $\dot{x} = \sin(x)$, $x(0) = \pi/2$.

8. Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be Lipschitz continuous and let $y(t)$ be a (nonconstant) periodic solution to $\dot{x} = f(x)$. Suppose that $x(t)$ is another solution such that $x(0)$ lies inside the region enclosed by $y(t)$. Prove that $x(t)$ exists for all $t \in \mathbb{R}$.

9. Let φ_t, ψ_t be globally defined flows on $\mathbb{R}^n$ and define $\rho_t(x) = \varphi_t(\psi_t(x))$.

(a) Suppose the flows φ_t and ψ_t commute – that is, $\varphi_s \circ \psi_t = \psi_t \circ \varphi_s$ for all s, t. Show that ρ_t is a flow.

(b) Give an example of non-commuting flows φ_t, ψ_t for which ρ_t is *not* a flow.

10. Show that $L = x^2 + y^2$ is a strict Lyapunov function for the planar system

$$\dot{x} = -x - y^2$$
$$\dot{y} = -y - x^2$$

on a neighbourhood of the origin. What is the maximum r such that the origin attracts the entire disc of radius r?