Math 6324 — Ordinary Differential Equations I — Fall 2012

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PROBLEM SET 3

Due *5pm Tuesday Nov. 20.* You must give complete justification for all answers in order to receive full credit.

1. Consider the non-linear system of ODEs

$$\begin{aligned} \dot{x}_1 &= -x_1, \\ \dot{x}_2 &= -x_2 + x_1^2, \\ \dot{x}_3 &= x_3 + x_2^2, \end{aligned} \tag{1}$$

which has an equilibrium point at $x = 0$.

(a) Write (1) as $\dot{x} = Ax + F(x)$, where $A \in \mathbb{M}_{3\times 3}$ and $DF(0) = 0$. Write the linear part as $A = P \oplus Q$ where $P\colon E^s \to E^s$ and $Q\colon E^u \to E^u$. Write the non-linear part as $F = F^s + F^u$ where $F^s\colon \mathbb{R}^3 \to E^s$ and $F^u\colon \mathbb{R}^3 \to E^u$.

(b) Show that the unstable subspace E^u is invariant for (1), so that in particular the unstable manifold W^u is just the unstable subspace.

(c) Recall that the stable manifold is the graph of a function $\psi\colon E^s \to E^u$ whose value $\psi(a)$ at $a \in E^s$ is given by $x^u(0)$, where x is the unique fixed point of the operator

$$(\mathcal{P}x)(t) = e^{Pt}a + \int_0^t e^{P(t-s)}F^s(x(s))\,ds - \int_t^\infty e^{Q(t-s)}F^u(x(s))\,ds.$$

Given a curve $x(t) = (x_1(t), x_2(t), x_3(t)) \to 0$, write down an explicit expression for $(\mathcal{P}x)(t)$ in the setting of (1) by using P, Q, F^s, F^u from part (a).

(d) The fixed point of $\mathcal{P}$ can be found as the limit of the curves $\mathcal{P}^n x$ for some starting curve $x\colon [0,\infty) \to \mathbb{R}^n$. For the starting curve $x(t) = e^{Pt}a$, which is the solution of the linear system, compute $\mathcal{P}^n x$ for $n = 1, 2, 3$. Observe that $\mathcal{P}^2(x)$ is the fixed point of $\mathcal{P}$ and use this to write down a formula for the stable manifold W^s.

2. The Lorenz equation in $\mathbb{R}^3$ is given by

$$\dot{x} = f(x) = \begin{pmatrix} x_2 - x_1 \\ \mu x_1 - x_2 - x_1 x_3 \\ x_1 x_2 - x_3 \end{pmatrix}$$

for a parameter $\mu > 0$. The origin is always an equilibrium point. At what value of μ do two new equilibrium points appear? Find the eigenvalues at the nonzero equilibrium points as a function of μ and describe the stability of these points.

3. Let $\varphi_t \colon U \to U$ and $\psi_t \colon V \to V$ be flows (here $U, V \subset \mathbb{R}^n$ are open). We say that the flows are topologically conjugate if there exists a homeomorphism $h \colon U \to V$ such that $h \circ \varphi_t = \psi_t \circ h$ for all $t \in \mathbb{R}$. If h is a C^1 diffeomorphism, we say the flows are C^1-conjugate.

(a) Suppose φ_t and ψ_t are topologically conjugate. Show that h maps equilibria of φ_t to equilibria of ψ_t, and similarly for periodic orbits. Need the period of corresponding periodic orbits be the same? Either prove or give a counterexample.

(b) Suppose φ_t and ψ_t are C^1-conjugate. Show that if x_0 and $h(x_0)$ are equilibria for φ_t, ψ_t respectively, then the linearisations have the same eigenvalues.

4. Consider the non-linear system of ODEs

$$\begin{aligned} \dot{x}_1 &= -x_1, \\ \dot{x}_2 &= -x_2 + x_1^2, \\ \dot{x}_3 &= x_3 + x_1^2, \end{aligned} \tag{2}$$

which has a hyperbolic equilibrium point at $x = 0$. The Hartman–Grobman theorem states that there is a topological conjugacy H between (2) and its linearisation on a neighbourhood of the origin.

(a) Solve the system (2) explicitly and write down a formula for the time-1 map φ_1 and its inverse φ_{-1}.

(b) Let L be the time-1 map of the linearised system, and let $L^s = L|_{E^s}$, $L^u = L|_{E^u}$, so that $L = L^s \oplus L^u$. Write $H = (H^s, H^u)$, where $H^s \colon \mathbb{R}^3 \to E^s$ and $H^u \colon \mathbb{R}^3 \to E^u$. Then the conjugacy H_0 between φ_1 and L is obtained as the fixed point of the operator $\mathcal{P}$ given by

$$(\mathcal{P}H)(x) = (L^s H^s(\varphi_{-1}(x)), (L^u)^{-1} H^u(\varphi_1(x))).$$

Given a function $H(x) = (H_1(x), H_2(x), H_3(x))$ on $\mathbb{R}^3$, write an explicit formula for $(\mathcal{P}H)(x_1, x_2, x_3)$ using the time-1 map you computed in part (a).

(c) Let $I \colon \mathbb{R}^3 \to \mathbb{R}^3$ be the identity map, and show that $\mathcal{P}^n I$ converges on $\mathbb{R}^3$ to a homeomorphism $H_0 \colon \mathbb{R}^3 \to \mathbb{R}^3$. Show that H_0 is a topological conjugacy between φ_1 and L.

(d) Use H_0 to find a topological conjugacy H between (2) and its linearisation on all of $\mathbb{R}^3$.