Math 6324, Differential Equations - Final exam review sheet

Linear theory.

- Basic definitions: vector space, norm, metric, topology
  - All norms are equivalent in finite dimensions
  - Operator norm on linear transformations/matrices
- Matrix exponential, basic properties
  - Formula for solution of arbitrary linear autonomous homogeneous first-order ODEs in terms of matrix exponentials
- Use of Jordan normal form to compute matrix exponentials
  - Complexification, semisimple and nilpotent matrices
  - Generalised eigenvectors, Jordan chains
  - Jordan normal form: description and existence
  - Results on matrices (linear transformations) using Jordan normal form (semisimple-nilpotent decomposition): Cayley–Hamilton theorem, properties of trace and determinant
- Higher order ODEs
  - Reduction to first order ODEs, solution through Jordan normal form
  - Analysis via operators on function spaces
- Non-homogeneous ODEs
  - Solution via variation of constants
- Stability of fixed point at 0
  - Spectral data: use of real parts of eigenvalues to determine if 0 is hyperbolic, a sink, or a source
  - Lyapunov functions for sinks

Nonlinear theory: general results.

- Different regularities of vector field: continuous, locally Lipschitz, Lipschitz, C^1
- Peano's existence theorem: existence of solutions for continuous vector field
  - Examples for non-uniqueness (leaky bucket)
- Picard–Lindelöf theorem: local existence and uniqueness for locally Lipschitz vector field
  - Picard iteration
  - Banach fixed point theorem
  - Local nature of solution, possibility of finite-time blowup
- Gronwall's lemma, global uniqueness, maximal interval of existence
- Dependence on initial conditions and parameters
  - Continuous dependence via Gronwall's lemma (locally Lipschitz vector field)
  - C^1 dependence via fundamental matrix solution to $\dot{v} = A(t)v$ for $A(t) = Df(x(t))$ (requires a C^1 vector field)
  - Properties of fundamental matrix solution
- Flow associated to an ODE/vector field, group law $\phi_{t+s} = \phi_t \circ \phi_s$

Nonlinear theory: tools for analysis of specific systems.

- Stability of equilibrium points
  - Lyapunov stability, asymptotic stability, exponential stability, instability
  - Use of linearisation to determine stability
    - Sinks (all eigenvalues have negative real part) are exponentially stable
    - Eigenvalue with positive real part implies unstable
  - Lyapunov functions, use to determine stability, asymptotic stability, size of basin of attraction
- Local behaviour near hyperbolic fixed points
  - Stable manifold theorem (Hadamard–Perron), local and global stable and unstable manifolds
  - Topological equivalence, topological conjugacy, C^1-conjugacy
    - Preservation of equilibrium points, periodic orbits, periods, eigenvalues under various notions of equivalent/conjugacy
  - Hartman–Grobman theorem on topological conjugacy to linearisation in neighbourhood of hyperbolic fixed point
    - Conjugacy can be made C^1 if vector field is C^2
- Stability of periodic orbits
  - Poincaré return map
  - Floquet theory, characteristic exponents, characteristic multipliers
- Bifurcations
  - Structural stability
  - Local bifurcations (changes in stability of a fixed point or periodic orbit)
  - Global bifurcations (such as changes in behaviour of homoclinic/heteroclinic orbits)
- Asymptotic behaviour, α- and ω-limit sets
  - Basic properties (closed and invariant, also non-empty, compact, and connected if trajectory contained in bounded region)
  - Positively and negatively invariant sets (forward- and backward-invariant), identifying with Lyapunov-type function, using to determine location of limit sets
  - Possible limit sets in $\mathbb{R}^2$, Poincaré–Bendixson theorems
    - Local sections, flow boxes, heteroclinic cycles
  - Comparison to behaviour in $\mathbb{R}^n$ for $n \geq 3$