Math 2331
Linear Algebra
Test 1 - Chapters 1 and 2
Practice Problems

Use the following vectors for questions 1-7

$$u = \begin{pmatrix} 1 \\ -2 \\ 4 \\ 2 \end{pmatrix} \qquad v = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad w = \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$$

1. Find $u \bullet w$ $= -1/2$

2. Find $u \bullet v$ $= 3$

3. Find $-2u + 4w$ $= \begin{pmatrix} -2 \\ 4 \\ -8 \\ -4 \end{pmatrix} + \begin{pmatrix} -2 \\ 2 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} -4 \\ 6 \\ -6 \\ -6 \end{pmatrix}$

4. Find $-v - w$ $= \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{1}{2} \\ -1/2 \\ -1/2 \\ 1/2 \end{pmatrix} = \begin{pmatrix} 3/2 \\ -1/2 \\ -3/2 \\ 1/2 \end{pmatrix}$

5. Are v and w orthogonal? $v \cdot w = 1$ No

$$||v|| = \sqrt{1+0+1+0} = \sqrt{2}$$

6. Find the length of each vector.

$$||u|| = \sqrt{u \cdot u} = \sqrt{25} = 5 \qquad ||w|| = \sqrt{\tfrac{1}{4}+\tfrac{1}{4}+\tfrac{1}{4}+\tfrac{1}{4}} = 1$$

7. State a unit vector in the direction of u.

$$\frac{u}{||u||} = \frac{1}{5}u = \begin{pmatrix} 1/5 \\ -2/5 \\ 4/5 \\ 2/5 \end{pmatrix}$$

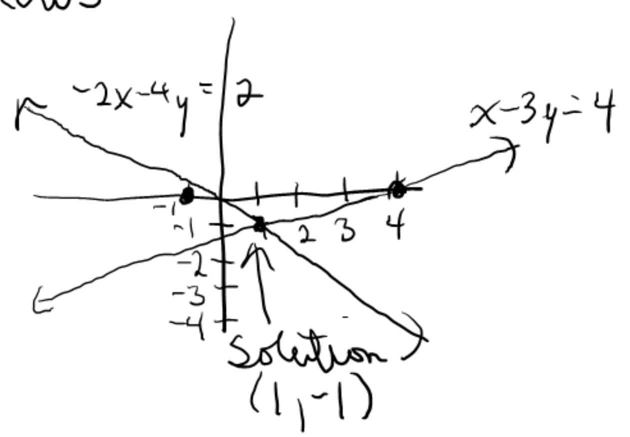
Answer questions 8-12 for each of the following systems:

$$\begin{aligned} x - 3y &= 4 \\ -2x - 4y &= 2 \end{aligned} \qquad \begin{aligned} x - y &= -1 \\ x + y &= 5 \end{aligned} \qquad \begin{aligned} x + 5y &= 5 \\ -x + 3y &= -5 \end{aligned}$$

8. Sketch the ROW PICTURE of the solution to the system.
9. Sketch the COLUMN PICTURE of the solution to the system.
10. State the system as a matrix equation.
11. State the elimination matrix E that transforms the system to upper triangular.
12. Solve by elimination and back substitution.

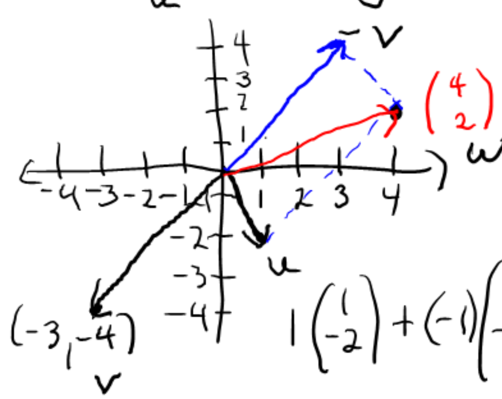
$$\begin{aligned} x - 3y &= 4 \\ -2x - 4y &= 2 \end{aligned}$$

ROWS



COLUMN

$$x \underbrace{\begin{bmatrix} 1 \\ -2 \end{bmatrix}}_{u} + y \overbrace{\begin{bmatrix} -3 \\ -4 \end{bmatrix}}^{v} = \overbrace{\begin{bmatrix} 4 \\ 2 \end{bmatrix}}^{w}$$



$$1\begin{pmatrix} 1 \\ -2 \end{pmatrix} + (-1)\begin{pmatrix} -3 \\ -4 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -3 \\ -2 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

$$E = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}\left(\begin{array}{cc|c} 1 & -3 & 4 \\ -2 & -4 & 2 \end{array}\right)$$

$$= \left(\begin{array}{cc|c} 1 & -3 & 4 \\ 0 & -10 & 10 \end{array}\right)$$

$$-10y = 10 \qquad x - 3(-1) = 4$$

$$y = -1 \qquad x = 1$$

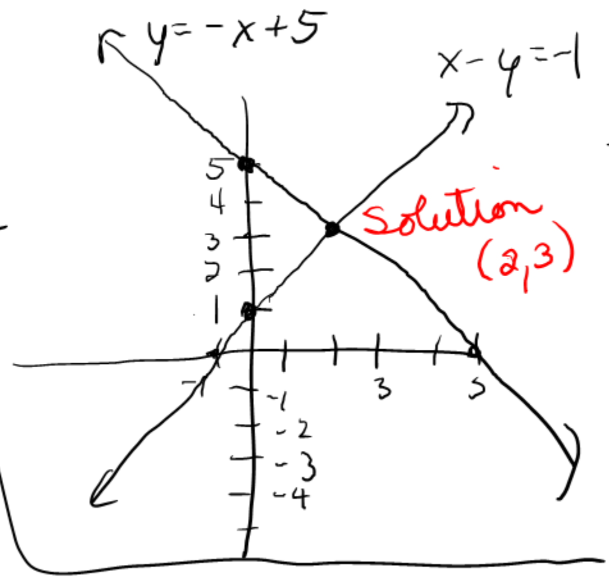
$x - y = -1$

$x + y = 5$

ROWS

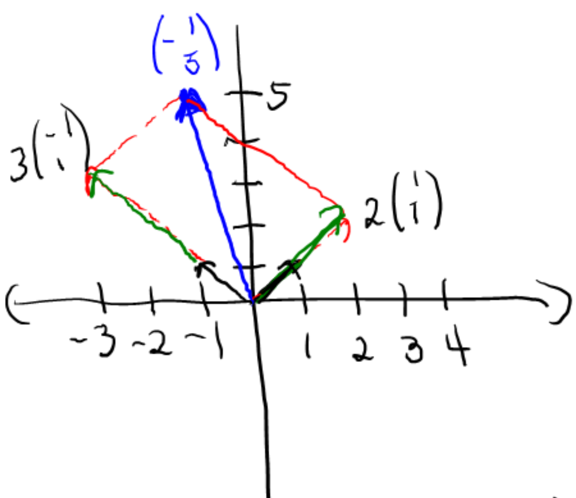
$$y = x + 1$$

$$y = x - 5$$



COLUMN

$$x\begin{pmatrix} 1 \\ 1 \end{pmatrix} + y\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$$



$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$$

ROW2 − ROW1 $\quad E = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$

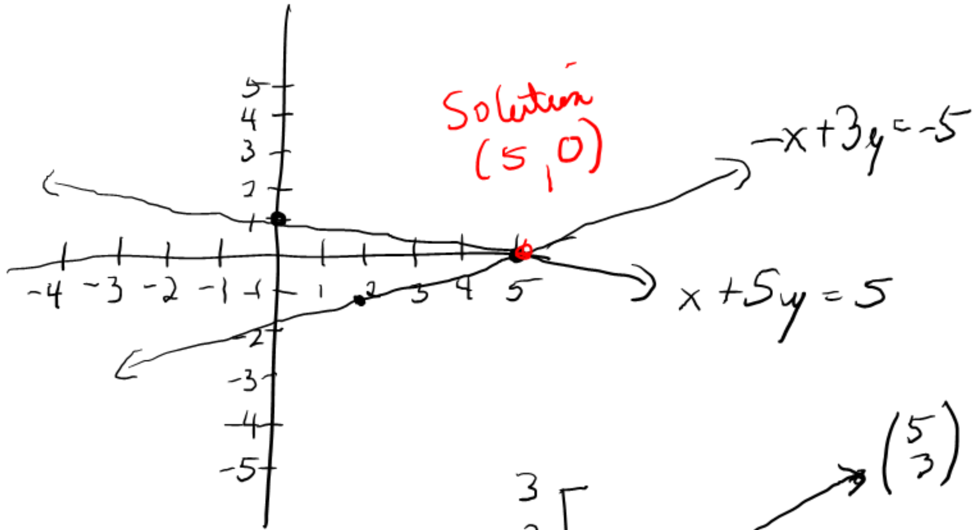
$$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\left(\begin{array}{cc|c} 1 & -1 & -1 \\ 1 & 1 & 5 \end{array}\right) = \left(\begin{array}{cc|c} 1 & -1 & -1 \\ 0 & 2 & 6 \end{array}\right)$$

$$2y = 6 \qquad x - 3 = -1$$

$$y = 3 \qquad x = 2$$

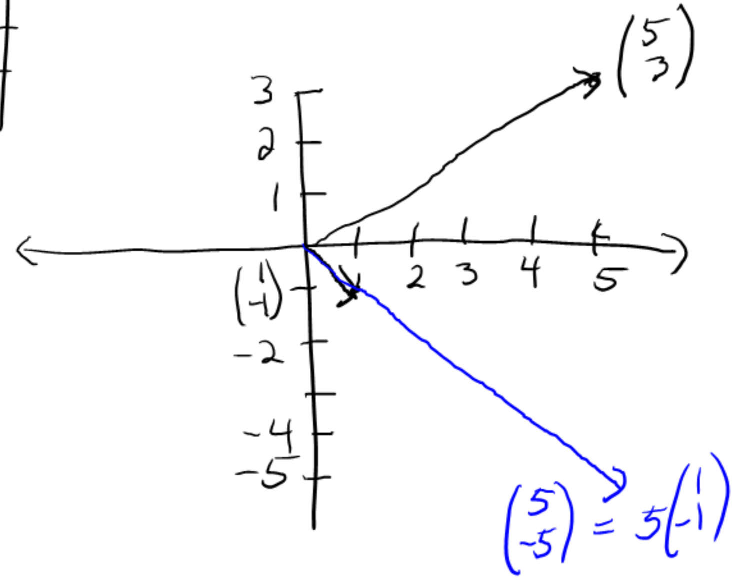
$$x+5y=5$$
$$-x+3y=-5$$

ROW



COLUMN

$$x\begin{pmatrix}1\\-1\end{pmatrix}+y\begin{pmatrix}5\\3\end{pmatrix}=\begin{pmatrix}5\\-5\end{pmatrix}$$



$$\begin{pmatrix}1&5\\-1&3\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}5\\-5\end{pmatrix}$$

$$E=\begin{pmatrix}1&0\\1&1\end{pmatrix}$$ ROW2 + ROW 1

$$\begin{pmatrix}1&0\\1&1\end{pmatrix}\left(\begin{array}{cc|c}1&5&5\\-1&3&-5\end{array}\right)=\left(\begin{array}{cc|c}1&5&5\\0&8&0\end{array}\right)$$

$$8y=0 \qquad x=5$$
$$y=0$$

State the LU and LDU factorization of each of the following matrices.

13. $A=\begin{pmatrix}1 & 3\\ 4 & 5\end{pmatrix} = \underset{L}{\begin{pmatrix}1 & 0\\ 4 & 1\end{pmatrix}}\underset{U}{\begin{pmatrix}1 & 3\\ 0 & -7\end{pmatrix}} = \underset{L}{\begin{pmatrix}1 & 0\\ 4 & 1\end{pmatrix}}\underset{D}{\begin{pmatrix}1 & 0\\ 0 & -7\end{pmatrix}}\underset{U}{\begin{pmatrix}1 & 3\\ 0 & 1\end{pmatrix}}$

14. $B=\begin{pmatrix}1 & 2 & 3\\ 2 & 3 & 0\\ 3 & 0 & 4\end{pmatrix} = \overset{L}{\begin{pmatrix}1 & 0 & 0\\ 2 & 1 & 0\\ 3 & -6 & 1\end{pmatrix}}\overset{U}{\begin{pmatrix}1 & 2 & 3\\ 0 & -1 & -6\\ 0 & 0 & -41\end{pmatrix}}$

$$= \underset{L}{\begin{pmatrix}1 & 0 & 0\\ 2 & 1 & 0\\ 3 & -6 & 0\end{pmatrix}}\underset{D}{\begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -41\end{pmatrix}}\underset{U}{\begin{pmatrix}1 & 2 & 3\\ 0 & 1 & 6\\ 0 & 0 & 1\end{pmatrix}}$$

15. $C=\begin{pmatrix}2 & 1 & 0 & 0\\ 1 & 2 & 1 & 0\\ 0 & 1 & 2 & 1\\ 0 & 0 & 1 & 2\end{pmatrix} = \begin{pmatrix}1 & 0 & 0 & 0\\ \frac{1}{2} & 1 & 0 & 0\\ 0 & \frac{2}{3} & 1 & 0\\ 0 & 0 & \frac{3}{4} & 1\end{pmatrix}\begin{pmatrix}2 & 1 & 0 & 0\\ 0 & \frac{3}{2} & 1 & 0\\ 0 & 0 & \frac{4}{3} & 1\\ 0 & 0 & 0 & \frac{5}{4}\end{pmatrix}$

$$= \begin{pmatrix}1 & 0 & 0 & 0\\ 1/2 & 1 & 0 & 0\\ 0 & 2/3 & 1 & 0\\ 0 & 0 & 3/4 & 1\end{pmatrix}\begin{pmatrix}\frac{2}{1} & 0 & 0 & 0\\ 0 & \frac{3}{2} & 0 & 0\\ 0 & 0 & \frac{4}{3} & 0\\ 0 & 0 & 0 & \frac{5}{4}\end{pmatrix}\begin{pmatrix}1 & \frac{1}{2} & 0 & 0\\ 0 & 1 & \frac{2}{3} & 0\\ 0 & 0 & 1 & \frac{3}{4}\\ 0 & 0 & 0 & 1\end{pmatrix}$$

Find the inverse, if it exists, of each of the following matrices.

16. $A = \begin{pmatrix} 1 & -1 & 3 \\ 4 & 0 & 5 \\ 2 & -1 & -3 \end{pmatrix}$

$A^{-1} = \begin{pmatrix} -5/29 & 6/29 & 5/29 \\ -22/29 & 9/29 & -7/29 \\ 4/29 & 1/29 & -4/29 \end{pmatrix}$

17. $M = \begin{pmatrix} -4 & 9 \\ -2 & 4 \end{pmatrix}$

$ad - bc = 2$

$M^{-1} = \frac{1}{2}\begin{pmatrix} 4 & -9 \\ 2 & -4 \end{pmatrix} = \begin{pmatrix} 2 & -9/2 \\ 1 & -2 \end{pmatrix}$

18. $B = \begin{pmatrix} 1 & 4 \\ 3 & 12 \end{pmatrix}$

$ad - bc = 0$

B is NOT INVERTIBLE

19. $A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$

$A^{-1} = \begin{pmatrix} \frac{3}{4} & -\frac{1}{2} & \frac{1}{4} \\ -\frac{1}{2} & 1 & -1/2 \\ \frac{1}{4} & -\frac{1}{2} & \frac{3}{4} \end{pmatrix}$

20. State the transpose of matrices in problems 16-19

#19 $A^{-1} = A$

#18 $A = \begin{pmatrix} 1 & 4 \\ 3 & 12 \end{pmatrix}$ $A^T = \begin{pmatrix} 1 & 3 \\ 4 & 12 \end{pmatrix}$

#17 $M = \begin{pmatrix} -4 & 9 \\ -2 & 4 \end{pmatrix}$ $M^T = \begin{pmatrix} -4 & -2 \\ 9 & 4 \end{pmatrix}$

#16 $A = \begin{pmatrix} 1 & -1 & 3 \\ 4 & 0 & 5 \\ 2 & -1 & -3 \end{pmatrix}$ $A^T = \begin{pmatrix} 1 & 4 & 2 \\ -1 & 0 & -1 \\ 3 & 5 & -3 \end{pmatrix}$

21. Given the matrices in problems 13-19, which are symmetric?

#14, #15, #20 are symmetric

22. State the 4x4 permutation matrix P that switches row 1 and row 3. What is the inverse of P?

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad P^{-1} = P^T = P$$

Given the following matrices and vectors,

$$A = \begin{pmatrix} 1 & -1 & 3 \\ 4 & 0 & 5 \\ 2 & -1 & -3 \end{pmatrix} \qquad M = \begin{pmatrix} 1 & 2 & 3 \\ -4 & 1 & 7 \end{pmatrix} \qquad P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$Q = \begin{pmatrix} 1 & 3 \\ -2 & 1 \\ 0 & 5 \end{pmatrix} \qquad b = \begin{pmatrix} 2 \\ -3 \\ 0 \end{pmatrix} \qquad c = \begin{pmatrix} 5 \\ 4 \end{pmatrix} \qquad x = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

23. Multiply

Ax = $\begin{pmatrix} 1 & -1 & 3 \\ 4 & 0 & 5 \\ 2 & -1 & -3 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 9 \\ -2 \end{pmatrix}$

Ab = $\begin{pmatrix} 1 & -1 & 3 \\ 4 & 0 & 5 \\ 2 & -1 & -3 \end{pmatrix}\begin{pmatrix} 2 \\ -3 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ 8 \\ 7 \end{pmatrix}$

24. Multiply

Mc

$\underset{2\times 3}{M} \quad \underset{2\times 1}{c}$ Mc NOT DEFINED

Mb $\begin{pmatrix} 1 & 2 & 3 \\ -4 & 1 & 7 \end{pmatrix}\begin{pmatrix} 2 \\ -3 \\ 0 \end{pmatrix} = \begin{pmatrix} -4 \\ -11 \end{pmatrix}$

25. Multiply

AM

$$\underset{3\times 3}{\begin{pmatrix} 1 & -1 & 3 \\ 4 & 0 & 5 \\ 2 & -1 & -3 \end{pmatrix}} \underset{2\times 3}{\begin{pmatrix} 1 & 2 & 3 \\ -4 & 1 & 7 \end{pmatrix}}$$ product not defined

MA

$$\begin{pmatrix} 1 & 2 & 3 \\ -4 & 1 & 7 \end{pmatrix} \begin{pmatrix} 1 & -1 & 3 \\ 4 & 0 & 5 \\ 2 & -1 & -3 \end{pmatrix} = \begin{pmatrix} 15 & -4 & 4 \\ 14 & -3 & -28 \end{pmatrix}$$

26. Multiply

QM

$$\underset{3\times 2}{\begin{pmatrix} 1 & 3 \\ -2 & 1 \\ 0 & 5 \end{pmatrix}} \underset{2\times 3}{\begin{pmatrix} 1 & 2 & 3 \\ -4 & 1 & 7 \end{pmatrix}} = \begin{pmatrix} -11 & 5 & 24 \\ -6 & -3 & 1 \\ -20 & 5 & 35 \end{pmatrix}$$

PQ

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ -2 & 1 \\ 0 & 5 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 1 & 3 \\ 0 & 5 \end{pmatrix}$$

27. Find the transpose of A, M and b

$$A^T = \begin{pmatrix} 1 & 4 & 2 \\ -1 & 0 & -1 \\ 3 & 5 & -3 \end{pmatrix} \qquad M^T = \begin{pmatrix} 1 & -4 \\ 2 & 1 \\ 3 & 7 \end{pmatrix} \qquad b^T = \begin{pmatrix} 2 & -3 & 0 \end{pmatrix}$$

28. Find the transpose of PQ and MP

$$(PQ)^T = \begin{pmatrix} -2 & 1 & 0 \\ 1 & 3 & 5 \end{pmatrix} = Q^T P^T$$

$$M = \begin{pmatrix} 1 & 2 & 3 \\ -4 & 1 & 7 \end{pmatrix} \qquad P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$(MP)^T = P^T M^T = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -4 \\ 2 & 1 \\ 3 & 7 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & -4 \\ 3 & 7 \end{pmatrix}$$

Solve the following systems by elimination and back substitution:

29. $\begin{pmatrix} 3 & 2 & 0 & 0 \\ 4 & 5 & 0 & 0 \\ 0 & 0 & 6 & 5 \\ 0 & 0 & 7 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 4 \\ 1 \end{pmatrix}$

$R2 - \frac{4}{3}R1$

$R4 - \frac{7}{6}R3$

$$\left(\begin{array}{cccc|c} 3 & 2 & 0 & 0 & 1 \\ 0 & 7/3 & 0 & 0 & 5/3 \\ 0 & 0 & 6 & 5 & 4 \\ 0 & 0 & 0 & \frac{1}{6} & -\frac{22}{6} \end{array}\right)$$

$y = 5/7$

$t = -22$

$$3x + 2\left(\frac{5}{7}\right) = 1$$

$$3x = 1 - \frac{10}{7} = -3/7 \qquad x = -1/7$$

$$6z + 5(-22) = 4$$

$$6z = 114 \qquad z = 19$$

30. $\begin{aligned} x+2y+2z&=1 \\ 4x+8y+9z&=3 \\ 3y+2z&=1 \end{aligned}$

$$\left(\begin{array}{ccc|c} 1 & 2 & 2 & 1 \\ 4 & 8 & 9 & 3 \\ 0 & 3 & 2 & 1 \end{array}\right)$$

$$R2-4R1 \quad \left(\begin{array}{ccc|c} 1 & 2 & 2 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 3 & 2 & 1 \end{array}\right)$$

$$\text{Switch } 2 \,\&\, 3 \quad \left(\begin{array}{ccc|c} 1 & 2 & 2 & 1 \\ 0 & 3 & 2 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

$$\begin{array}{l} R1-2R3 \\ R2-2R3 \end{array} \quad \left(\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & 3 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right) \quad \tfrac{1}{3}R_2 \left(\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

$$R1-2R2 \quad \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right) \quad \begin{aligned} x&=1 \\ y&=1 \\ z&=-1 \end{aligned}$$