First day of class: went over the course webpage briefly, told class to Please download and read everything that has been put up on this 3321 website. Familiarize yourself with all class policies, and the upcoming due dates. Also mentioned I had sent emails to everyones UH email address, you will need to keep watching that address since we often will communicate by email. Please check you got my emails.

Chapter 1. Intro to differential equations

1.1: Basic Terminology

• A differential equation (DE) is an equation involving derivatives $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \cdots$.

• An ordinary differential equation (ODE) is an equation involving ordinary derivatives $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \cdots$.

• A partial differential equation (PDE) is an equation involving partial derivatives. We will not deal with PDE's in this course really.

• Partial derivatives: for example $\frac{\partial f}{\partial x}$, or $\frac{\partial u}{\partial z}$, or $\frac{\partial^2 u}{\partial z \partial x}$.

• For example, $\frac{\partial}{\partial z}$ means that one only differentiates with respect to z, treating other (independent) variables as constants.

• For example,

$$\frac{\partial}{\partial x}(2x^2y^3) = 2x^2\frac{\partial}{\partial x}(y^3) = 2x^2 \cdot 3y^2 = 6x^2y^2$$

• Example 1. The equation

$$x^2y'' - 2x\,y' + 6\,y = x^3\cos x$$

is a 2nd order ordinary differential equation (ODE). It is 2nd order because the highest derivative is a 2nd derivative. Familiarize yourself with the following ways of writing the same equation:

$$x^2\frac{d^2y}{dx^2} - 2x\,\frac{dy}{dx} + 6\,y = x^3\cos x$$

$$x^2D^2y - 2x\,Dy + 6\,y = x^3\cos x$$

where D stands for the first derivative, D^2 for the 2nd derivative, etc,

$$(x^2D^2 - 2x\,D + 6I)\,y = x^3\cos x$$

$$L\,y = x^3\cos x$$

where $L = x^2D^2 - 2x\,D + 6I$ (called a *differential operator*).

These are 5 ways of writing the same equation.

• Example 2. The equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2x^2y^3$$

is a 2nd order partial differential equation (PDE).

• Example 3. The equation

$$\frac{3}{t^2}\frac{d^2x}{dt^2} - \frac{5}{t}\frac{dx}{dt} + 7x = \sin\, t$$

is a 2nd order ordinary differential equation (ODE). The dependent variable this time is x, the independent variable is t. Familiarize yourself with the following ways of writing the same equation:

$$\frac{3}{t^2}D^2x - \frac{5}{t}Dx + 7x = \sin\, t$$

$$(\frac{3}{t^2}D^2 - \frac{5}{t}D + 7I)x = \sin\, t$$

and

$$Lx = \sin\, t, \qquad L = \frac{3}{t^2}D^2 - \frac{5}{t}D + 7I.$$

• More generally the nth order ordinary differential equation (ODE) is of the form

$$F(x, y, y', y'', \cdots, y^{(n)}) = 0.$$

In Example 1 above, $F(x, y, y', y'') = x^2y'' - 2x\,y' + 6\,y - x^3 \cos x$.

1.2: Families of Solutions; General Solution; Particular Solution

• A *solution* or *particular solution* to an ODE is a function which, after you have taken its derivatives and plugged these into the DE, makes that equation true (or makes it true on an interval or domain of numbers that you care about). We say that the equation is 'valid' or that the equation 'holds', when it is true.

• Example. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$ has, as one solution (or as a particular solution) the function $y = e^x$. This is because after you have taken its derivatives $y' = e^x$ and $y'' = e^x$, and have plugged these into the DE, we get

$$e^x - 2e^x + e^x = 0$$

which is true. Another solution (or particular solution) is $y = xe^x$. This is because after you have taken its derivatives $y' = e^x + xe^x$ and $y'' = e^x + e^x + xe^x = 2e^x + xe^x$, and plug these into the DE, we get

$$2e^x + xe^x - 2(e^x + xe^x) + xe^x = 0$$

which is true.

• An *integral curve* is the graph of a (particular) solution to an ODE. For example the exponential graph $y = e^x$ is an integral curve for the DE in the last example.

• The *general solution* of any ODE is a complete list of all the solutions to the ODE. We will see later that the general solution to the ODE $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$ in the last example is

$$y = Ce^x + Dxe^x,$$

where C and D are arbitrary constants. This is also called a *two parameter family of solutions.* Here C, D are the parameters, and there are two of them. A parameter is a 'constant that can change'. Saying that $y = Ce^x + Dxe^x$ is the general solution is saying that every solution to this ODE is of the form $Ce^x + Dxe^x$, for some values of C, D.

• One of the main goals of this course is finding the general solution to an nth order ODE. Usually, but not always, this general solution will be an n-parameter family of solutions, so there will be n arbitrary constants. Sometimes we will find an n-parameter family of solutions to the nth order ODE, but there will also be other solutions not in this n-parameter family. The latter are called *singular solutions*. So the general solution will sometimes be the n-parameter family of solutions, together with one or more singular solutions.

• Example. Find the set of all solutions (which is the same as the general solution) to $y'' = -x + 2\sin x$.

Solution: Integrating once, $y' = -\frac{1}{2}x^2 - 2\cos x + C$. Integrating again, we get

$$y = -\frac{1}{6}x^3 - 2\sin x + Cx + D.$$

This is the general solution. It is also the two parameter family of solutions. There are no singular solutions.

• Example. Find a one parameter family of solutions, and find the singular solutions, if any, and the general solution, to the ODE $\frac{dy}{dx} = 4x(y-1)^{\frac{1}{2}}$, for $y \geq 1$.

Solution: We use the *separation of variables technique*. That is, put all y's on the left side and all x's on the right: Divide both sides by $(y-1)^{\frac{1}{2}}$ and multiply both sides by dx:

$$\frac{dy}{(y-1)^{\frac{1}{2}}} = 4x\,dx$$

Add an integral sign to both sides:

$$\int \frac{dy}{(y-1)^{\frac{1}{2}}} = \int 4x\,dx = 2x^2 + C.$$

To do the first integral we do a u-substitution, let $u = y-1$, then $du = dy$, and the integral becomes

$$\int \frac{dy}{(y-1)^{\frac{1}{2}}} = \int u^{-\frac{1}{2}} du = 2u^{\frac{1}{2}} = 2(y-1)^{\frac{1}{2}}.$$

Thus we now have

$$2(y-1)^{\frac{1}{2}} = 2x^2 + C$$

or (dividing by 2)

$$(y-1)^{\frac{1}{2}} = x^2 + C$$

so that

$$y - 1 = (x^2 + C)^2$$

or

$$y = (x^2 + C)^2 + 1.$$

This is the one parameter family of solutions. But there is one more solution which is not in this family: $y = 1$. If $y = 1$ then $\frac{dy}{dx} = 0$, and $4x(y-1)^{\frac{1}{2}} = 0$, so $y = 1$ is a solution to our ODE. But $y = 1$ is not in the family $(x^2 + C)^2 + 1$, since these are all polynomials of degree 4, not constants. So $y = 1$ is a SINGULAR SOLUTION.

The method of separation of variables which we used here is not at fault here, it did not cause us to miss the singular solutions. The fault here that caused us to miss the singular solutions was that we were a bit careless. In the first step, where we divided by $(y-1)^{\frac{1}{2}}$, we should have remembered that one cannot divide by zero. So we should ask when $(y-1)^{\frac{1}{2}} = 0$, and treat that as a separate case. This happens exactly when $y - 1 = 0$, or $y = 1$. One then needs to check if this is a solution or not. In this case it is, so $y = 1$ is a singular solution. Apart from this one carelessness our method was fine, so there is only one singular solution. The general solution is the one parameter family of solutions $y = (x^2 + C)^2 + 1$ together with the singular solution $y = 1$.

• On the online quiz there will be many questions asking you to "Identify the differential equation solved by" a certain function. Its a multiple choice, so you can just take the derivatives of this function, and plug into each of the given answers and see which one works. Similarly, some questions say "The function $\cdots$ is the general solution of" then lists five answers. Here $\cdots$ is the function you will see there. Again you can just take the derivatives of this function, and plug into each of the given answers and see which one works.

• Why are DE's (differential equations) important?

• Recall that $\frac{dy}{dx}$ is the rate of change of y with respect to x. That is, how fast y is changing. Recall that $\frac{d^2y}{dx^2}$ is the rate of change of the rate of change of y. That is, it is the rate of change of $\frac{dy}{dx}$. This is *acceleration* or *decceleration*. Similarly, $\frac{d^3y}{dx^3}$ is the rate of change of the acceleration, and so on.

• DE's (differential equations) are expressions of how these rates of change are related to each other, which happens all the time in life, physics, finance, biology, etc. Most of the phenomena studied in the sciences and engineering involve processes that change with time.

• Example from Calculus 2: "Natural" growth or "Exponential growth" (or decay).
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$$\frac{dy}{dx} = K\,y.$$

• This is a first order ODE, which says that quantity y is growing at a rate proportional to the size of y.

• In Calculus 2, or in other classes like finance, one talks about how important this equation is. Its solution, from those courses is

$$y = Ce^{Kx}$$

Here C is an arbitary constant. This is the *general solution* to the ODE $\frac{dy}{dx} = Ky$.

• We shall meet later other examples/applications. For example, the position of an object, suspended by a spring, oscillating up and down. Newtons Second Law of Motion (F = ma) combined with Hookes Law (the restoring force of a spring is proportional to the displacement of the object) results in the 2nd order ODE $y'' + k^2y = 0$.

• On the online quiz there will be some questions asking you to find differential equation that a certain function satisfies. It might also say "Find the differential equation of the given family". Or if the family is given as a series of curves (graphs), one could ask "Find the differential equation of the following family of curves" (or integral curves).

This is really the reverse procedure to solving a differential equation (where we start with the DE and end up with a certain function or family).

To show how to do this, we work an example or two.

• Example 1. Find the DE of the family $y^2 = Cx^3 + 3$. (This may also be asked as: "Give the differential equation that has $y^2 = Cx^3 + 3$ as its general solution.")

Solution. Strategy: Find y', and $y'', \cdots$ etc if needed, and then try find a formula connecting them. By implicit differentiation, $2yy' = 3Cx^2$. So $C = \frac{2yy'}{3x^2}$, and plugging this into the original ODE gives

$$y^2 = \frac{2yy'}{3x^2}x^3 + 3 = \frac{2xyy'}{3} + 3,$$

or, solving the latter for y' we get

$$y' = \frac{3(y^2 - 3)}{2xy}.$$

• Example 2. Find the DE that has $y = C_1 + C_2x^3$ as its general solution.

Solution. Similar. Note $y' = 3C_2x^2$ and $y'' = 6C_2x$. Solving for C_2 we get $C_2 = \frac{y'}{3x^2}$ and $C_2 = \frac{y''}{6x}$. So $\frac{y''}{6x} = \frac{y'}{3x^2}$, or $y'' = \frac{2}{x}y$.

1.3: Initial Conditions; Initial-Value Problems

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TO BE ADDED