Lab quiz 10

1. Is the series $\sum_{n=1}^{\infty} \frac{(-1)^n 2n^7 - 6}{8n^{11}+3n^5-2}$

   (A) Absolutely Convergent

   (B) Conditionally Convergent

   (C) Divergent

   Solution:

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n 2n^7 - 6}{8n^{11}+3n^5-2} \right| = \sum_{n=1}^{\infty} \frac{2n^7 - 6}{8n^{11}+3n^5-2}$$

$$\text{Let } \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{2n^7 - 6}{8n^{11}+3n^5-2} \text{ and } \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^4}$$

We see,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2n^{11} - 6n^4}{8n^{11}+3n^5-2} = \frac{1}{4} > 0$$

Now $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^4}$ is convergent by p-series test as p=4 > 1, hence $\sum_{n=1}^{\infty} a_n$ is convergent by limit comparison test. Thus the series is absolutely convergent.

2. Is the series $\sum_{n=1}^{\infty} \frac{(-1)^n 3n^5+3n^2}{9n^6+8n^5+n}$

(A) Absolutely Convergent

(B) Conditionally Convergent

(C) Divergent

Solution:

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n 3n^5 + 3n^2}{9n^6 + 8n^5 + n} \right| = \sum_{n=1}^{\infty} \frac{3n^5 + 3n^2}{9n^6 + 8n^5 + n}$$

$$\text{Let } \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{3n^5 + 3n^2}{9n^6 + 8n^5 + n} \text{ and } \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$$

We see,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{3n^6 + 3n^3}{9n^6 + 8n^5 + n} = \frac{1}{3} > 0$$

Now $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$ is divergent by p-series test as p=1, hence $\sum_{n=1}^{\infty} a_n$ is divergent by limit comparison test. Thus the series is not absolutely convergent.

Now $a_n = \frac{3n^5+3n^2}{9n^6+8n^5+n}$ is (i) continuous, (ii) monotonically decreasing and (iii) $\lim_{n\to\infty} \frac{3n^5+3n^2}{9n^6+8n^5+n} = 0$ so by alternating series test the series is convergent.

Hence the series is conditionally convergent.

3. Is the series $\sum_{n=1}^{\infty} \frac{(-1)^n (n-6)^2}{(n+2)(n+4)}$

(A) Absolutely Convergent

(B) Conditionally Convergent

(C) Divergent

Solution:

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n (n-6)^2}{(n+2)(n+4)} \right| = \sum_{n=1}^{\infty} \frac{(n-6)^2}{(n+2)(n+4)}$$

$$\text{We see } \lim_{n\to\infty} \frac{(n-6)^2}{(n+2)(n+4)} = \lim_{n\to\infty} \frac{n^2+36-12n}{n^2+6n+8} = 1 \not\to 0$$

Hence it is divergent by basic divergence test. Thus the series is not absolutely convergent

Now $a_n = \frac{(n-6)^2}{(n+2)(n+4)}$ is (i) continuous, (ii) monotonically decreasing but (iii) $\lim_{n\to\infty} \frac{(n-6)^2}{(n+2)(n+4)} = 1 \neq 0$ so we cannot apply alternating series test.

Hence the series is divergent.

4. Is the series $\sum\limits_{n=1}^{\infty} \frac{(-1)^n(arctan(2n))}{4n^2+1}$

(A) Absolutely Convergent

(B) Conditionally Convergent

(C) Divergent

Solution:

Method 1:

$$\sum_{n=1}^{\infty}\left|\frac{(-1)^n(arctan(2n))}{4n^2+1}\right| = \sum_{n=1}^{\infty}\frac{arctan(2n)}{4n^2+1} \leq \sum_{n=1}^{\infty}\frac{\frac{\pi}{2}}{4n^2+1} \leq \frac{\pi}{2}\sum_{n=1}^{\infty}\frac{1}{4n^2}$$

Now $\sum\limits_{n=1}^{\infty}\frac{1}{4n^2}$ is convergent by p-series test as p=2.
Hence the series is convergent by basic comparison test, thus the series is absolutely convergent.

Method 2:

$$\sum_{n=1}^{\infty}\left|\frac{(-1)^n(arctan(2n))}{4n^2+1}\right| = \sum_{n=1}^{\infty}\frac{arctan(2n)}{4n^2+1}$$

Now $a_n = \frac{arctan(2n)}{4n^2+1}$ is (i) continuous, (ii) positive and (iii) decreasing so we can apply integral test.

$$\int_1^{\infty}\frac{arctan(2x)}{4x^2+1}dx = \lim_{b\to\infty}\int_1^{b}\frac{arctan(2x)}{4x^2+1}dx$$

$$\int\frac{arctan(2x)}{4x^2+1}dx = \int udu \quad (u = tan^{-1}2x \Rightarrow du = \frac{dx}{4x^2+1})$$
$$= \frac{u^2}{2} + c = \frac{(tan^{-1}2x)^2}{2} + c$$

Therefore,

$$\int_1^{\infty}\frac{arctan(2x)}{4x^2+1}dx = \lim_{b\to\infty}\frac{(tan^{-1}2x)^2}{2}\Big|_1^b$$
$$= \lim_{b\to\infty}\frac{(tan^{-1}2b)^2}{2} - \frac{(tan^{-1}2)^2}{2}$$
$$= \frac{(\frac{\pi}{2})^2}{2} - \frac{(tan^{-1}2)^2}{2} < \infty$$

Hence the series is convergent by integral test, thus the series is absolutely convergent.

5. Find the interval of convergence of $\sum_{n=0}^{\infty} \frac{(x-1)^n}{3^n}$

(A) $[-2, 4]$

(B) $[-3, 3]$

(C) $(\frac{2}{3}, \frac{4}{3})$

(D) $(-2, 4)$

(E) $(-3, 3)$

Solution:

$$\lim_{n\to\infty} \left| \left(\frac{(x-1)^n}{3^n} \right)^{\frac{1}{n}} \right| = \lim_{n\to\infty} \left| \frac{x-1}{3} \right| = \frac{|x-1|}{3}$$

So the radius of convergence is 3.
To be convergent by root test we must have

$$\frac{|x-1|}{3} < 1 \Rightarrow |x-1| < 3 \Rightarrow -3 < x-1 < 3 \Rightarrow -2 < x < 4$$

At $x = -2$ we have $\displaystyle\sum_{n=0}^{\infty} \frac{(-2-1)^n}{3^n} = \sum_{n=0}^{\infty} \frac{(-3)^n}{3^n} = \sum_{n=0}^{\infty} \frac{(-1)^n 3^n}{3^n} = \sum_{n=0}^{\infty} (-1)^n \Rightarrow$ Divergent

At $x = 4$ we have $\displaystyle\sum_{n=0}^{\infty} \frac{(4-1)^n}{3^n} = \sum_{n=0}^{\infty} \frac{(3)^n}{3^n} = \sum_{n=0}^{\infty} (1)^n \Rightarrow$ Divergent

Therefore, the interval of convergence is $(-2, 4)$.