Lab quiz 11

1. The radius of convergence of the power series $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n^n}$ is:

   (A) 0
   (B) 1
   (C) 3
   (D) ∞
   (E) None of the above

   Solution:

   $$\lim_{n\to\infty} \left| \left(\frac{(x-3)^n}{n^n} \right)^{\frac{1}{n}} \right| = \lim_{n\to\infty} \left| \frac{(x-3)}{n} \right| = 0$$

   Thus radius of convergence is ∞.

2. Interval of convergence of the power series $\sum\limits_{n=1}^{\infty} \frac{(-1)^n(x-5)^n}{4^n n}$ is:

(A) $[1, 9]$

(B) $[1, 9)$

(C) $(1, 9]$

(D) $(1, 9)$

(E) None of the above

Solution:

$$\lim_{n\to\infty} \left| \left(\frac{(-1)^n(x-5)^n}{4^n n} \right)^{\frac{1}{n}} \right| = \lim_{n\to\infty} \left| \frac{(-1)(x-5)}{4n^{\frac{1}{n}}} \right| = \left| \frac{x-5}{4} \right|$$

To be convergent we need

$$\left| \frac{x-5}{4} \right| < 1 \Rightarrow |x-5| < 4 \Rightarrow -4 < x-5 < 4 \Rightarrow 1 < x < 9$$

For $x = 1$ we have

$$\sum_{n=1}^{\infty} \frac{(-1)^n(1-5)^n}{4^n n} = \sum_{n=1}^{\infty} \frac{(-1)^n(-1.4)^n}{4^n n} = \sum_{n=1}^{\infty} \frac{(-1)^n(-1)^n(4)^n}{4^n n} = \sum_{n=1}^{\infty} \frac{(-1)^{2n}4^n}{4^n n} = \sum_{n=1}^{\infty} \frac{1}{n}$$

Now $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ is divergent by p-series test as p=1.

For $x = 9$ we have

$$\sum_{n=1}^{\infty} \frac{(-1)^n(9-5)^n}{4^n n} = \sum_{n=1}^{\infty} \frac{(-1)^n(4)^n}{4^n n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

Now $a_n = \frac{1}{n}$ is continuous, monotonically decreasing and $\lim\limits_{n\to\infty} \frac{1}{n} = 0$. So $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{n}$ is convergent by alternating series test.

Hence, the interval of convergence is $(1, 9]$

(If you see $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{n}$ is not absolutely convergent as $\left| \sum\limits_{n=1}^{\infty} \frac{(-1)^n}{n} \right| = \sum\limits_{n=1}^{\infty} \frac{1}{n}$ is divergent.)

3. The Taylor series in x for $f(x) = x^2 ln(1+x)$ is

(A) $\sum \frac{(-1)^{n+1}x^{n+2}}{n!}$

(B) $\sum \frac{(-1)^{n+1}x^{n+1}}{n}$

(C) $\sum \frac{(-1)^{n+1}x^{n+2}}{n}$

(D) $\sum \frac{(-1)^{n+1}x^{n+1}}{n!}$

(E) $\sum \frac{(-1)^{n+1}x^{n+2}}{(n+1)!}$

Solution:

$$ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{n}.$$

$$\text{So } x^2 ln(1+x) = x^2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^{n+2}}{n}$$

4. Write following function as a power series using sigma notation: $f(x) = \frac{6}{1+6x}$

(A) $\sum 6^{k+1}x^k, \quad \text{if } |6x| < 1$
(B) $\sum (-1)^k 6^{k+1}x^k, \quad \text{if } |6x| < 1$
(C) $\sum (-1)^k 6^k x^k, \quad \text{if } |6x| < 1$
(D) $\sum 6^{k+1}x^k, \quad \text{if } |6x| < 1$
(E) $\sum (-1)^k 6x^k, \quad \text{if } |6x| < 1$

[Extra, not in lab quiz: Using the above problem write the power series for $g(x) = ln(1+6x)$]

Solution:

$$\begin{aligned} f(x) &= \frac{6}{1+6x} \\ &= \frac{6}{1-(-6x)} \\ &= \sum_{n=0}^{\infty} 6.(-6x)^n, \quad \text{if } |6x| < 1 \\ &= \sum_{n=0}^{\infty} 6.(-1)^n 6^n x^n, \quad \text{if } |6x| < 1 \\ &= \sum_{n=0}^{\infty} (-1)^n 6^{n+1} x^n, \quad \text{if } |6x| < 1 \end{aligned}$$

Extra part:

Observe that,

$$\int \frac{6}{1+6x} dx = ln(1+6x) + c$$

$$\int \sum_{n=0}^{\infty} (-1)^n 6^{n+1} x^n dx = ln(1+6x) + c, \quad \text{if } |6x| < 1$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n 6^{n+1} x^{n+1}}{n+1} = ln(1+6x) + c, \quad \text{if } |6x| < 1$$

For x=0 we have,

$$\sum_{n=0}^{\infty} \frac{(-1)^n 6^{n+1} 0^{n+1}}{n+1} = ln(1+6.0) + c \Rightarrow 0 = ln(1) + c \Rightarrow c = 0$$

Therefore,

$$\sum_{n=0}^{\infty} \frac{(-1)^n 6^{n+1} x^{n+1}}{n+1} = ln(1+6x), \quad \text{if } |6x| < 1$$

5. Using the Taylor polynomial for $f(x) = x^2cos(2x)$ centered at $x = 0$ the $f^4(0)$ is:

(A) 0

(B) 24

(C) -24

(D) 48

(E) -48

Solution:

$$cos(x) \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$$
$$cos(2x) \approx 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!}$$
$$x^2cos(2x) \approx x^2(1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!})$$
$$= x^2 - \frac{2^2x^4}{2!} + \frac{2^4x^6}{4!}$$

[For explanation only:
Observe that

$$f'(x) = 2x - \frac{2^2.4.x^3}{2!} + \frac{2^4.6x^5}{4!}$$
$$f''(x) = 2 - \frac{2^2.4.3x^2}{2!} + \frac{2^4.6.5x^4}{4!}$$
$$f^3(x) = -\frac{2^2.4.3.2x}{2!} + \frac{2^4.6.5.4.x^3}{4!}$$
$$f^4(x) = -\frac{2^2.4.3.2.1}{2!} + \frac{2^4.6.5.4.3x^2}{4!}$$
$$= -\frac{2^2.4!}{2!} + \frac{2^4.6.5.4.3x^2}{4!}$$

So all the powers of x less than 4 vanishes in $f^4(x)$. While the higher powers of x which are more than 4 survive. But the question ask for $f^4(x)$ at $x = 0$, so putting x=0 in $f^4(x)$ all the terms with x becomes 0.]

Therefore,

$$f^4(0) = -\frac{2^2.4!}{2!} = -\frac{4.24}{2} = -48$$