Lab quiz 8

1. If possible find the sum of the $\sum_{n=0}^{\infty} \frac{1-2^n}{3^{n+1}}$

   (A) $\frac{1}{2}$

   (B) $-\frac{1}{2}$

   (C) 3

   (D) $\frac{3}{2}$

   (E) Divergent

   Solution :

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{1-2^n}{3^{n+1}} &= \sum_{n=0}^{\infty} \frac{1-2^n}{3^n.3} \\
&= \frac{1}{3} \sum_{n=0}^{\infty} \left[\left(\frac{1}{3}\right)^n - \left(\frac{2}{3}\right)^n \right] \\
&= \frac{1}{3} \left[\frac{(\frac{1}{3})^0}{1-\frac{1}{3}} - \frac{(\frac{2}{3})^0}{1-\frac{2}{3}} \right] \\
&= \frac{1}{3} \left[\frac{1}{\frac{2}{3}} - \frac{1}{\frac{1}{3}} \right] \\
&= \frac{1}{2} - 1 \\
&= -\frac{1}{2}
\end{aligned}$$

2. The series $\sum_{n=1}^{\infty} \frac{ln(n^6)}{3n}$ is :

(A) Convergent

(B) Divergent

Solution:
Method 1

$$\sum_{n=1}^{\infty} \frac{ln(n^6)}{3n} = \sum_{n=1}^{\infty} \frac{6ln(n)}{3n} = 2\sum_{n=1}^{\infty} \frac{ln(n)}{n} \geq 2\sum_{n=1}^{\infty} \frac{1}{n}$$

Now $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent by p-series test as p=1. Hence, by basic divergence test $\sum_{n=1}^{\infty} \frac{ln(n^6)}{3n}$ is divergent.

Method 2

$a_n = \frac{ln(n^6)}{3n}$ is (i) continuous as the numerator and denominator are continuous, (ii) positive as numerator and denominator are positive for all $n \geq 1$ and (iii) decreasing (convince yourself!!!) so we can apply integral test.

$$\int_1^{\infty} \frac{ln(x^6)}{3x}dx = \lim_{b\to\infty}\int_1^b \frac{6ln(x)}{3x}dx = 2\lim_{b\to\infty}\int_1^b \frac{ln(x)}{x}dx$$

Now

$$\int \frac{ln(x)}{x}dx = \int udu \quad (u = lnx \implies du = \frac{dx}{x})$$
$$= \frac{u^2}{2} + c = \frac{(ln(n))^2}{2} + c$$

Hence $\lim_{b\to\infty}\int_1^b \frac{ln(x)}{x}dx = \lim_{b\to\infty} \frac{(ln(n))^2}{2}\Big|_1^b = \lim_{b\to\infty} \frac{(ln(b))^2}{2} - \frac{(ln(1))^2}{2} = \lim_{b\to\infty} \frac{(ln(b))^2}{2} =$ DNE

As the integral diverges therefore by integral test the series is divergent.

3. The series $\sum_{n=1}^{\infty} \left(\frac{n+2}{n+1}\right)^n$ is :

(A) Convergent

(B) Divergent

Solution:

(Check if you apply root test the result will be inconclusive)

$$\begin{aligned}\lim_{n\to\infty}\left(\frac{n+2}{n+1}\right)^n &= \lim_{n\to\infty}\left(\frac{n+1+1}{n+1}\right)^n\\ &= \lim_{n\to\infty}\left(\frac{n+1}{n+1}+\frac{1}{n+1}\right)^n\\ &= \lim_{n\to\infty}\left(1+\frac{1}{n+1}\right)^n\\ &= \lim_{m\to\infty}\left(1+\frac{1}{m}\right)^{m-1}\end{aligned}$$

(Let $m = n+1 \implies n = m-1$. And if $n \to \infty$ then $m \to \infty$)

$$\begin{aligned}&= \lim_{m\to\infty}\left(1+\frac{1}{m}\right)^m . \lim_{m\to\infty}\left(1+\frac{1}{m}\right)^{-1}\\ &= e.(1+0)^{-1}\\ &= e \not\to 0\end{aligned}$$

As the limit of the sequence does not go to 0 the series is divergent by basic divergence test.

4. The series $\sum\limits_{n=1}^{\infty} \frac{7n^{3/2}+2n}{3n^{5/2}+\sqrt{n}}$ is:

(A) Convergent

(B) Divergent

Solution:

Method 1:

Let $\sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty} \frac{7n^{3/2}+2n}{3n^{5/2}+\sqrt{n}}$ and $\sum\limits_{n=1}^{\infty} b_n = \sum\limits_{n=1}^{\infty} \frac{1}{n}$. (We get by $n^{\frac{3}{2}-\frac{5}{2}} = n^{-1}$)

Then $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \lim\limits_{n\to\infty} \frac{7n^{5/2}+2n^2}{3n^{5/2}+\sqrt{n}} = \frac{7}{3} > 0.$

Now $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ is divergent by p-series test as p=1. Hence by limit comparison test the series $\sum\limits_{n=1}^{\infty} a_n$ is divergent.

Method 2:

Convince yourself about each step of the inequality. We can increase a fraction if we increase the numerator or decrease the denominator.

$$\sum_{n=1}^{\infty} \frac{7n^{3/2}+2n}{3n^{5/2}+\sqrt{n}} \leq \sum_{n=1}^{\infty} \frac{7n^{3/2}+2n}{3n^{5/2}} \leq \sum_{n=1}^{\infty} \frac{7n^{3/2}+2n^{3/2}}{3n^{5/2}} = \sum_{n=1}^{\infty} \frac{5n^{3/2}}{3n^{5/2}} = \sum_{n=1}^{\infty} \frac{5}{3n}$$

Now $\sum\limits_{n=1}^{\infty} \frac{5}{3n}$ is divergent by p-series test as p=1 .
Hence the series is divergent by basic divergence test.

5. The series $\sum_{n=1}^{\infty} n^{5/n}$ is:

(A) Convergent

(B) Divergent

Solution:

$$\begin{aligned}\lim_{n\to\infty} n^{5/n}(\infty^0 \text{ form}) &= \lim_{n\to\infty} e^{ln(n^{5/n})}\\ &= \lim_{n\to\infty} e^{\frac{5}{n}ln(n)}\\ &= e^{5\lim_{n\to\infty}\frac{ln(n)}{n}}\end{aligned}$$

Now $\lim_{n\to\infty} \frac{ln(n)}{n}\left(\frac{\infty}{\infty} \text{ form, L'Hospital}\right) = \lim_{n\to\infty} \frac{\frac{\mathrm{d}}{\mathrm{d}n}ln(n)}{\frac{\mathrm{d}}{\mathrm{d}n}(n)} = \lim_{n\to\infty} \frac{1}{n.1} = 0$

Therefore $\lim_{n\to\infty} n^{5/n} = e^{5.0} = e^0 = 1 \not\to 0$

As the limit of the sequence does not go to zero the series is divergent by basic divergence test.